Notes on Groups and Representations

G. Jungman

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Part I

Theory

Chapter 1

Topological Groups

1.1 Invariant Measures

Remark. Many of the fundamental results on groups and their represenataions can be given in the general context of topological groups, without recourse to differential theory. This is, however, a difficult approach. Essentially the only tools available in the theory of topological groups are measure theory and C^*-algebras. This makes the landscape somewhat stark and alien. But topological group theory provides an important application of these tools, and we will begin here.

Definition. A topological group is a group which is also a topological space, such that the group operation $x, y \mapsto xy^{-1}$ is continuous.

Definition. Given $a \in G$, define the left translation map

$$\tau_a : G \longrightarrow G$$
$$\tau_a : g \mapsto ag$$

As a consequence of the continuity of the group operations, left translation is a homeomorphism.

Definition. Let $x, y \in G$, and let U be an open neighbourhood of the identity. We say that x, y are U-close if $x \in yU$. In this way, open nbhds. of the identity take the place of "δ" in ϵ, δ arguments, and the topological group effectively masquerades as a metric space.

Let H be a subgroup of G, and let G/H be the set of left cosets $\{xH : x \in G\}$. Let π be the canonical map

$$\pi : G \longrightarrow G/H.$$

Endow G/H with the topology such that $W \subset G/H$ is open if and only if $\pi^{-1}(W)$ is open in G. In general G/H is not Hausdorff. However, if H is closed then G/H will be Hausdorff. The converse is also true.

1.1. Theorem (Open Projection). *π is an open mapping (maps open sets to open sets).*

Proof. Let $W \subset G/H$ be open; $W = \pi(\pi^{-1}(W))$ and $\pi^{-1}(W)$ is open by assumption. Conversely let $V \subset G$ be open. Then $\pi^{-1}(\pi(V)) = VH$ is open in G. So $\pi(V)$ is open. □

Remark. Recall that a topological space is called locally compact if every point has a compact neighbourhood. Further recall that a Borel σ-algebra is the σ-algebra generated by the open sets of a topological space. Now we come to the single most important result in the theory of topological groups, which is the existence and uniqueness of a translation invariant measure.

Definition. Define a Haar functional to be a positive non-zero functional λ on $C_0(G)$, which is left-invariant

$$\lambda(\tau_g f) = \lambda(f), \quad \forall f \in C_0(G).$$

Definition. A positive measure μ on a locally compact space is called σ-regular when the following hold.

- If A is a Borel set, then $\mu(A) = \inf \mu(V)$, $A \subset V$.
- If K is compact, then $\mu(K) < \infty$.
- If A is Borel and it is a countable union of sets of finite measure [σ-finite], then $\mu(A) = \sup \mu(K)$, with K compact, $K \subset A$.

Definition. Define a Haar measure on a locally compact group G to be a positive σ-regular measure on the σ-algebra of Borel sets which is non-zero on any non-empty open set, and which is left-invariant,

$$\mu(gA) = \mu(A), \quad A \text{ measurable}, A \subset G, g \in G.$$

Remark. To prove uniqueness of Haar measure, it is useful to reduce the problem from G to a certain distinguished subgroup of G. The following lemma accomplishes this reduction.

1.2. Lemma (Distinguished Subgroup). *Let G be a locally compact group. Then there exists a subgroup $H \subset G$ which is both open and closed, and which is a countable union of compact sets.*

Proof. Let K be a compact neighbourhood of $1 \in G$ with $K = K^{-1}$. Define $K^n = K \cdot K \cdots K$ n-times. These are compact nbhds. of 1, and they form an increasing sequence $\cdots \subseteq K^n \subseteq K^{n+1} \subseteq \cdots$. Let $H = K^\infty = \bigcup_{n>0} K^n$. By construction then H is a countable union of compact sets. By construction H is a subgroup as well.

Let $x \in H$, so $x \in K^n$ for some n. Therefore $xK \subset K^{n+1} \subset H$, and so a neighbourhood of x is contained in H. Therefore H is open.

Now H is the complement of a union of all cosets which are not identically equal to H. But all cosets of H are open, so H is the complement of an open set, therefore closed. □

Remark. The distinguished subgroup that is constructed here should be thought of as "all elements which can be reached from a neighbourhood of the identity with countably many group operations." Because it is constructed from a countable number of compact sets, it will have simple measure theoretic properties. On the other hand, it is large enough that we can work on it instead of on all of G.

1.3. Lemma. *Let H be the σ-finite subgroup constructed above. Uniqueness of Haar measure on H implies uniqueness on G, up to an overall constant of normalization.*

Proof. Use the local compactness of G to write G as a union of a countable number of translates of H,

$$G = \bigcup_{i \in I} x_i H.$$

Suppose that μ is a Haar measure on G. By the left-translation invariance of μ, the $x_i H$ are all equivalent as measure spaces. Each $x_i H$ is certainly σ-finite, by construction. If $A \subset G$ is an arbitrary measurable set then we can write

$$A = \bigcup_i A_i, \quad A_i \subset x_i H, \qquad A_i \bigcap A_j = \emptyset \quad i \neq j.$$

Now suppose that we have proven the uniqueness of Haar measure on H. Then compute as follows.

$$\begin{aligned} \mu_1(\bigcup A_i) &= \sum \mu_1(A_i) \\ &= \sum \mu_1(x_i^{-1} A_i \subset H) \\ &= \sum c\mu_2(x_i^{-1} A_i) \\ &= c\mu_2(\bigcup A_i). \end{aligned}$$

This demonstrates that the measures are unique up to an overall normalization, which is what we wished to show. □

Remark. We will associate the measure μ with the functional $d\mu$ in the usual way. That this is a bijective correspondence is not difficult [Lan83, p.429]. It is simplest to prove the result using the associated functionals rather than the measures.

1.4. Theorem (Haar Uniqueness). *Let μ_1 and μ_2 be Haar measures on G. Then there exists a number $c > 0$ such that $\mu_1(\cdot) = c\mu_2(\cdot)$.*

Proof. As shown above, it suffices to prove the result for σ-finite G. Therefore assume that G is σ-finite. This allows us to use the Fubini theorem freely. Let

$$r(f) = \frac{\int f \, d\mu_1}{\int f \, d\mu_2}$$

for non-zero positive $f \in C_0(G)$. Let $\psi(x)$ be a function supported in a neighbourhood of $1 \in G$, and let $h(x) = N\psi(x)\psi(x^{-1})$. If ψ is chosen positive and the constant N is chosen appropriately, h will satisfy

$$h(x) = h(x^{-1}), \qquad \int h \, d\mu_2 = 1.$$

Let the support be chosen inside a compact neighbourhood K of $1 \in G$. Now compute, using the left-invariance and the symmetry of h,

$$\begin{aligned}
\int hd\mu_1 \int fd\mu_2 - \int hd\mu_2 \int fd\mu_1 &= \iint [h(x)f(y) - h(y)f(x)]\, d\mu_1(x)d\mu_2(y) \\
&= \iint \left[h(y^{-1}x)f(y) - h(y)f(yx)\right] d\mu_1(x)d\mu_2(y) \\
&= \iint \left[h(x^{-1}y)f(y) - h(y)f(yx)\right] d\mu_1(x)d\mu_2(y) \\
&= \iint h(x^{-1}xy)f(xy)d\mu_1(x)d\mu_2(xy) - \iint h(y)f(yx)d\mu_1(x)d\mu_2(y) \\
&= \iint h(y)\,[f(xy) - f(yx)]\, d\mu_1(x)d\mu_2(y).
\end{aligned}$$

Choose K small enough that, for a given $\epsilon > 0$,

$$|f(xy) - f(yx)| < \epsilon \quad \forall x \in G, y \in K.$$

Let $S = \operatorname{supp}(f)$, which is compact by hypothesis. Then the function $x \mapsto f(xy) - f(yx)$ has support in a compact set of bounded μ_1-measure, $SK^{-1} \cup K^{-1}S$. By positivity of h,

$$\left| \int hd\mu_1 \int fd\mu_2 - \int hd\mu_2 \int fd\mu_1 \right| \leq \epsilon C_f \int hd\mu_2 = \epsilon C_f,$$

where C_f is a constant depending only on f. Therefore

$$\left| \int hd\mu_1 - \int hd\mu_2 \frac{\int fd\mu_1}{\int fd\mu_2} \right| \leq \epsilon C_f,$$

Therefore $|\int hd\mu_1 - r(f)| \leq \epsilon C_f$, and $\lim_{K\to\{1\}} \int h_K d\mu_1 = r(f)$. Therefore $r(f)$ is independent of f, and therefore $d\mu_1(\cdot) = cd\mu_2(\cdot)$. □

Remark. Unfortunately, the existence proof for Haar measure is not particularly useful. It proceeds by first constructing an approximately additive left-invariant functional for each function supported near the identity in G. Then the set of such functions is topologized and a compactness property is used to assert the existence of a strictly additive functional. No constructive formulae appear. See [Lan83, p.431] for the details.

Remark. It is a theorem of Weil that local compactness of G is *necessary as well as sufficient* for the existence of a non-zero left-invariant measure.

Remark. In all of the above constructions, "left-invariant" can be replaced with "right-invariant". However, the relation between the two views is not absolutely trivial.

1.5. Theorem (Modular Function). *Given a locally compact group G, there exists a group homomorphism $\Delta_G : G \longrightarrow \mathbb{R}^*$ such that, if μ_L and μ_R are left- and right-invariant measures, then*

- $d\mu_R(xy) = \Delta_G(x)d\mu_R(y)$
- $d\mu_L(xy) = \Delta_G(y)^{-1}d\mu_L(y)$
- $d\mu_R(x) = c\Delta_G(x)d\mu_L(x)$

- $d\mu_L(x^{-1}) = \Delta_G(x)d\mu_L(x)$
- $d\mu_R(x^{-1}) = \Delta_G(x)^{-1}d\mu_R(x)$

Furthermore, we have the explicit formula

$$\Delta_G(x) = \frac{\int f(x^{-1}y)d\mu_R(y)}{\int f(y)d\mu_R(y)}.$$

Proof. All are explicit computations. The only technical fact is the continuity of Δ_G, which follows from the explicit formula. Some of the computations are as follows. Let $d\mu_L$ be a left Haar measure. Define a second one by

$$d\mu_L^{(z)}(x) = d\mu_L(xz), \quad z \in G.$$

Clearly $d\mu_L^{(z)}$ is also a left Haar measure. By the uniqueness theorem we must have $d\mu_L^{(z)} = \Delta_G(z)^{-1}d\mu_L$, where $\Delta_G(z)$ is a number depending only on z. Applying the definition twice gives

$$\begin{aligned} d\mu_L(xyz) &= \Delta_G(y)^{-1}\Delta_G(z)^{-1}d\mu_L(x) \\ &= \Delta_G(yz)^{-1}d\mu_L(x), \\ \rightarrow \quad \Delta_G(xy) &= \Delta_G(x)\Delta_G(y). \end{aligned}$$

Similarly

$$\begin{aligned} d\mu_L(xyy^{-1}) = d\mu_L(x) &= \Delta_G(y)^{-1}\Delta_G(y^{-1})^{-1}d\mu_L(x), \\ \rightarrow \quad \Delta_G(y^{-1}) &= \Delta_G(y)^{-1}. \end{aligned}$$

Therefore $\Delta_G : G \longrightarrow \mathbb{R}$ is a group homomorphism. Note also that if we define a measure $d\mu(x) = \Delta_G(x)d\mu_L(x)$, then

$$\begin{aligned} d\mu(xy) &= \Delta_G(xy)d\mu_L(xy) \\ &= \Delta_G(xy)\Delta_G(y)^{-1}d\mu_L(x) \\ &= \Delta_G(x)d\mu_L(x) \\ &= d\mu(x). \end{aligned}$$

Therefore $d\mu(x)$ is right-invariant, $d\mu_R(x) = c\Delta_G(x)d\mu_L(x)$. So the function $\Delta_G(x)$ relates left-invariant and right-invariant measures. It is an important invariant of the group G. $\square$

Definition. Define G to be unimodular when $\Delta_G(x) = 1$. Note that this implies the existence of a two-sided-invariant measure.

Remark. We can now exhibit some Haar measures as examples. In general this is the way that we will come about Haar measures; they are usually just guessed, and the uniqueness theorem assumes a central role.

- Let $G = \mathbb{R}$ considered as an additive abelian group. Then Lebesgue measure is a Haar measure. The group is abelian, therefore it is trivially unimodular.

- Let $G = \mathrm{SL}_2(\mathbb{C})$. Represent the matrices as

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1.$$

  Then the following measure is left- and right-invariant,

$$d\mu(g) = |\delta|^{-2} d\beta d\overline{\beta} d\gamma d\overline{\gamma} d\delta d\overline{\delta}.$$

- Let $G = \mathrm{SL}_2(\mathbb{R})$. Represent the matrices as above. Then the following measure is left- and right-invariant,

$$d\mu(g) = \frac{d\alpha\, d\beta\, d\gamma\, d\delta}{(\alpha\delta - \beta\gamma)^{-2}}.$$

- Let $G = \mathrm{GL}_n(\mathbb{R})$. Represent the matrices as

$$M = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}.$$

  Then the following measure is left- and right-invariant,

$$d\mu(M) = \frac{dm_{11} dm_{12} \cdots dm_{nn}}{|\det M|^n}.$$

- Let $G = \mathbb{R}^*$, the multiplicative group of positive reals. Then the following measure is left- and right-invariant,

$$d\mu = dx/x.$$

- Let $G = \mathrm{Affine}(\mathbb{R})$ be the group of affine maps of the line, $x \mapsto ax + b$, $a \in \mathbb{R}^*$, $b \in \mathbb{R}$. Consider the measure

$$d\mu_R = \frac{da}{a} db,$$

  where elements of G are represented as $g = (a, b)$. Then the right-action is given by

$$\rho_{(a_0,b_0)} : (a, b) \mapsto (a', b') = (aa_0, ab_0 + b).$$

  So the measure is right-invariant. However, the group is not unimodular. We have

$$d\mu_L = \frac{da}{a^2} db, \quad \Delta_G(a, b) = a.$$

1.6. Lemma (Compact Unimodularity). *Let G be a compact topological group. Then G is unimodular.*

Proof. By the previous explicit formula, $\Delta_G : G \longrightarrow \mathbb{R}^*$ is a continuous group homomorphism. Being a continuous map, it maps compact sets into compact sets. Therefore the image of G in $\mathbb{R}^*$ is compact. It is also a subgroup of $\mathbb{R}^*$. However, it is easy to see that the only compact subgroup of $\mathbb{R}^*$ is $\{1\}$. Therefore $\Delta_G = 1$. □

Definition. We will now consider topological spaces which have degrees of symmetry described by topological groups. These are spaces which admit actions by such a group. Let X be a topological space. Define a G-action on X to be a map

$$\alpha : G \times X \longrightarrow X$$

such that

- α is continuous.
- $\alpha(g_1 g_2, x) = \alpha(g_1, \alpha(g_2, x))$
- $\alpha(1, x) = x$.

When a distinction is necessary, such a map will be called a left G-action on X. Define α to act transitively if for all $x, y \in X$ there exists $g \in G$ such that $\alpha(g, x) = y$. Define the stability group of $x \in X$ to be $\{g \in G \;:\; \alpha(g, x) = x\}$.

1.2 Coset Spaces

Remark. We have already considered the cosets G/H for closed subgroups H. These were seen to be Hausdorff spaces, and they were topologized such that the canonical projection was continuous. These are clearly topological spaces which admit a G-action. We will now show that these are the only examples.

1.7. Theorem. *Let G be a locally compact topological group acting transitively on a locally compact space X, by α. Let $x \in X$ be any point and let H be the stability group of x. Then G/H is homeomorphic to X, by the mapping $gH \mapsto \alpha(g, x)$.*

Proof. H is closed because it is the inverse image of a closed set under the map $\phi : g \mapsto \alpha(g, x)$, which is continuous by assumption.

Let $V \subset G$ be open, with $g \in V$. Let U be a neighbourhood of $1 \in G$ which is symmetric ($U^{-1} = U$), and which is sufficiently small that $gU^2 \subset V$.

G has a countable base, so there exists a sequence $\{g_n\}$ such that $G = \cup_n (g_n U)$. Let $X_n = \alpha(g_n U, x)$. Since the action is transitive, $X = \cup_n X_n$; each X_n is closed. By the Baire category theorem, one of the X_n must have a non-empty interior, say X_N.

Let $u \in G$ be such that $\alpha(u, x) \in X_N$ is a guaranteed interior point. Then $\alpha(u^{-1}U, x) \subset \alpha(U^2, x)$. So $\alpha(g, x)$ has an interior, which is covered by the interior of V, an so it is open. Our hypothesis was that V was open. Therefore ϕ is an open mapping. So ϕ is a homeomorphism. □

Remark. The above constructions can be carried out equally for right coset spaces and right-actions. We use the notation $H \backslash G$ for right coset spaces.

1.8. Theorem. *There exists an invariant measure μ on G/H if and only if $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$. This measure satisfies*

$$\int_G f(g)dg = \int_{G/H} d\mu(gH) \left[\int_H f(hg)dh \right].$$

Proof. See [BR77, p.130][Loo60]. □

Remark. In a precise sense, topological groups which look the same locally are all derived from a universal group which has a simple global structure.

1.9. Theorem (Universal Covering). *Let G_1 be arcwise-connected, locally connected, and locally simply-connected. Then there exists a unique simply-connected group $\tilde{G}$ with a* discrete *normal subgroup N_1 such that $G_1 \simeq \tilde{G}/N_1$. Furthermore, if G_2 is another such group, which is locally isomorphic to G_1, then there is another discrete normal subgroup N_2 such that $G_2 \simeq \tilde{G}/N_2$, for the same $\tilde{G}$.*

Proof. Most of the statements are true in the general context of covering spaces. Uniqueness of the universal covering is simple because no simply-connected space has a proper covering and any two coverings have a common covering. See, for example [Ger85]. The specific group-theoretical results are more difficult. See [Pon66]. □

Definition. The group $\tilde{G}$ is called the universal covering group of G_1, G_2.

Chapter 2

Extensions and Group Cohomology

Definition. Let G_0 be a normal subgroup of G. Then we have the following exact sequence

$$1 \to G_0 \xrightarrow{i} G \xrightarrow{\pi} G/G_0 \to 1.$$

Let $G_1 = G/G_0$. Then we say that this situation gives an extension of G_1 by G_0. Given two groups G_0, G_1, it is interesting to ask when such extensions exist.

Definition. The simplest type of extension is the semi-direct product. Since this is a familiar construction it is a good place to start. Let G_0, G_1 be groups. Let $\psi : G_1 \longrightarrow \mathrm{Aut}\,(G_0)$ be a representation of G_1 on $\mathrm{Aut}\,(G_0)$. Let $G = G_1 \times G_0$ be the Cartesian product of G_0 and G_1. We define a group operation on G by

$$(g_1, g_0) \cdot (g_1', g_0') = (g_1 g_1', [\psi^{-1}(g_1')g_0]g_0').$$

Associativity is the only hard part. It is a five or six line computation. Clearly the injections

$$\begin{aligned} i_0 &: G_0 \longrightarrow G \\ i_0 &: g_0 \mapsto (1, g_0) \\ \\ i_1 &: G_1 \longrightarrow G \\ i_1 &: g_1 \mapsto (1, g_1) \end{aligned}$$

are group homomorphisms. So we have an extension. This extension is called the semi-direct product of G_0 by G_1 with ψ; we write

$$G = G_0 \rtimes_\psi G_1.$$

Remark. Examples of semi-direct products are easy to come by. The simplest example is probably the affine motions of the line, a group we have seen before,

$$\begin{gathered} \mathrm{Affine}\,(\mathbb{R}) = \mathbb{R} \rtimes \mathbb{R}^* \\ (a, b) \equiv x \to ax + b \\ (a, b)(a', b') = (aa', ab' + b). \end{gathered}$$

Remark. The following lemma shows how more general extensions can be constructed.

2.1. Lemma (General Extensions). *Let two maps be given,*

$$\begin{aligned} \chi &: G_1 \times G_1 \longrightarrow G_0, \\ \psi &: G_1 \longrightarrow \mathrm{Aut}\,(G_0)\,, \end{aligned}$$

and require that they satisfy the conditions

- $\psi(g_1)\psi(g_1') = \psi(g_1 g_1')p(\chi(g_1, g_1'))$
- $\chi(g_1 g_1', g_1'')\psi(g_1'')^{-1} = \chi(g_1, g_1' g_1'')\chi(g_1', g_1'')$,

were $p : G_0 \longrightarrow \mathrm{Aut}\,(G_0)$ is the representation of G_0 on itself by conjugation, so $p(G_0) = \mathrm{Inn}\,(G_0)$, the inner automorphisms of G_0. When such maps exist satisfying these properties we have an extension of G_1 by G_0. Note that in general $\psi : G_1 \longrightarrow \mathrm{Aut}\,(G_0)$ is not required to be a representation.

Proof. The conditions for an extension can be verified from the conditions on the given maps. □

Definition. When $\psi : G_1 \longrightarrow \mathrm{Aut}\,(G_0)$ is a homomorphism, we say that the situation describes a central extension. In this case $p(\chi(g_1, g_1')) = 1$, therefore $\chi(g_1, g_1')^{-1} g_0 \chi(g_1, g_1') = g_0$ for all g_1, g_1', g_0. Therefore $\chi(G_1 \times G_1) \subseteq \mathcal{Z}\,(G_0)$, the center of G_0.

Remark. In general there may exist no such maps χ, ψ, and then no extensions of G_1 by G_0 will exist. When extensions exist they may not be unique.

Definition. Let G act on an abelian group A. Define a map c to be an n-dimensional cochain if

$$\begin{aligned} &c : G \times \cdots \times G \longrightarrow A \\ &c : (g_0, \ldots, g_n) \mapsto c(g_0, \ldots, g_n) \\ &(gg_0, \ldots, gg_n) = gc(g_0, \ldots, g_n). \end{aligned}$$

Definition. Define a coboundary operator by

$$dc(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i c(g_0, \ldots, \hat{g}_i, \ldots, g_{n+1}).$$

Definition. A cochain c is called a coboundary if $c = db$. It is called a cocycle if $dc = 0$.

2.2. Lemma. *The set of n-dimensional cochains forms a group $C^n(G, A)$. The set of n-dimensional cocycles forms a group $Z^n(G, A)$, containing the n-dimensional coboundaries as a subgroup, $B^n(G, A) \leq Z^n(G, A)$.*

Definition. Define the n-dimensional cohomology group to be the factor group

$$\mathrm{H}^n\,(G, A) = Z^n(G, A)/B^n(G, A).$$

When A has the extra structure of a ring or algebra, then $\mathrm{H}^n\,(G, A)$ is a also a (graded) ring or algebra.

Remark. Rather than deal with $c(g_0, \dots, g_n)$ explicitly, it is easier to compute with the representation $\tilde{c}(g_1, \dots, g_n) = c(1, g_1, g_1g_2, \dots, g_1g_2\cdots g_n)$. Then

$$\begin{aligned}
\tilde{d}c(h) &= h\tilde{c} - \tilde{c} \\
\tilde{d}c(h_1, h_2) &= h_1\tilde{c}(h_2) - \tilde{c}(h_1h_2) + \tilde{c}(h_1), \\
\tilde{d}c(h_1, h_2, h_3) &= h_1\tilde{c}(h_2, h_3) - \tilde{c}(h_1h_2, h_3) + \tilde{c}(h_1, h_2h_3) - \tilde{c}(h_1, h_2) \\
&\dots .
\end{aligned}$$

Remark. Now return to the discussion of general extensions. Given two groups, it is natural to ask when extensions of one by the other exist and to classify the extensions when they exist. In general there are obstructions to the existence of extensions. The following theorem characterizes those obstructions and the extensions.

2.3. Theorem. *If* $\mathrm{H}^3\left(G_1, \mathcal{Z}\left(G_0\right)\right) = 0$*, then there exist extensions of* G_1 *by* G_0*. Equivalence classes of central extensions of* G_1 *by* G_0 *are in one-to-one correspondence with elements of* $\mathrm{H}^2\left(G_1, \mathcal{Z}\left(G_0\right)\right)$.

Proof. Consider the map ψ. The first required property states that

$$\psi(g_1)\psi((g_1') = \psi(g_1g_1') \cdot p(\chi(g_1, g_1')).$$

Therefore $\psi(g_1)\psi(g_1')$ and $\psi(g_1g_1')$ differ by an element of $\mathrm{Im}\, p = \mathrm{Inn}\,(G_0)$. So they define the same element of the factor space $\mathrm{Aut}\,(G_0)/\mathrm{Inn}\,(G_0)$. The conjugation map $p : G_0 \longrightarrow G_0$ is an isomorphism $G_0/\ker p \simeq \mathrm{Inn}\,(G_0)$. The kernel is just the center ofG_0, so $\mathrm{Inn}\,(G_0) \simeq G_0/\mathcal{Z}\,(G_0)$. Therefore $\chi(g_1, g_1')$ is defined as an element of $G_0/\mathcal{Z}\,(G_0)$, i.e. defined up to an element of $\mathcal{Z}\,(G_0)$.

In general the second required property can fail to hold. However, the two sides can differ by at most an element of $\mathcal{Z}\,(G_0)$, call it $\omega(g_1, g_1', g_1'') \in \mathcal{Z}\,(G_0)$. This can be seen by applying p to both sides. So ω is a map $\omega : G_1 \times G_1 \times G_1 \longrightarrow \mathcal{Z}\,(G_0)$.

Furthermore, we have freedom in changing χ as long as we do not disturb the second property. Multiply that property by some $\beta : G_1 \times G_1 \longrightarrow \mathcal{Z}\,(G_0)$, and this alters ω by a factor

$$\beta(g_1g_1', g_1'')\beta(g_1, g_1')\beta(g_1', g_1'')^{-1}\beta(g_1, g_1'g_1'')^{-1}.$$

Finally, the map ω must satisfy

$$\omega(g_1, g_2, g_3)^{-1}\omega(g_1, g_2, g_3g_4)\omega(g_1, g_2g_3, g_4)^{-1} = \omega(g_2, g_3, g_4)\omega(g_1g_2, g_3, g_4)^{-1}.$$

These statements are more transparent in the additive notation. Trivial changes in ω are given by

$$\omega \to \omega + \beta(g_1g_2, g_3) + \beta(g_1, g_2) - \beta(g_2, g_3) - \beta(g_1, g_2g_3),$$

and the constraint on ω is

$$-\omega(g_1, g_2, g_3) + \omega(g_1, g_2, g_3g_4) - \omega(g_1, g_2g_3, g_4) - \omega(g_2, g_3, g_4) + \omega(g_1g_2, g_3, g_4) = 0.$$

This means $d\omega = 0$, where $\omega : G_1 \times G_1 \times G_1 \longrightarrow \mathcal{Z}\,(G_0)$ is regarded as a 3-cochain, i.e. as a representation of a 3-cochain in the form of a function of one less argument ($c \to \tilde{c}$). The trivial changes in ω are given by $\omega \to \omega + d\beta$.

Therefore, nontrivial obstructions to the second property are elements of $\mathrm{H}^3\left(G_1, \mathcal{Z}\left(G_0\right)\right)$. Nontrivial changes of extension are elements of $\mathrm{H}^2\left(G_1, \mathcal{Z}\left(G_0\right)\right)$. $\square$

Chapter 3

Representations of Topological Groups

3.1 Introduction

Definition. Let G be a locally compact separable topological group. Let $\mathcal{H}$ be a separable Hilbert space, and let ρ be a homomorphism of G into the set of bounded linear operators on $\mathcal{H}$ which is continuous for the strong operator topology.

$$\rho : G \longrightarrow \mathcal{B}(\mathcal{H})$$
$$\rho(xy) = \rho(x)\rho(y).$$

Then ρ is called a representation of G on $\mathcal{H}$.

Definition. A unitary representation of G is a representation into the set of unitary operators on $\mathcal{H}$.

Definition. Let $\mathcal{H} = L^2(G;\mu)$. The left regular representation is defined by

$$(\rho_a f)(g) = f(a^{-1}g), \quad f \in \mathcal{H}.$$

By invariance of μ each ρ_a is isometric and has domain equal to $\mathcal{H}$. So this representation is unitary. Strong continuity is easily proven by approximating $f \in \mathcal{H}$ by $C_0(G)$ functions.

Definition. Let ρ and ρ' be representations of G in $\mathcal{H}$ and $\mathcal{H}'$. Define an intertwining operator to be a bounded linear map $V : \mathcal{H} \longrightarrow \mathcal{H}'$ such that

$$V\rho_g = \rho'_g V, \quad \forall g \in G.$$

Denote the set of intertwining operators for ρ, ρ' by $\mathrm{Hom}_G(\rho, \rho')$. Generically $\mathrm{Hom}_G(\rho, \rho')$ is a linear space. When $\rho = \rho'$ then $\mathrm{Hom}_G(\rho, \rho)$ is an algebra.

Definition. Two representations are unitarily equivalent when there exists a unitary intertwiner for them. They are simply equivalent when an intertwiner of any type exists.

3.1. Theorem (Unitary Intertwiners). *Let ρ, ρ' be unitary representations which are equivalen-t.Then they are unitarily equivalent.*

Proof. By assumption there exists $V : \mathcal{H} \longrightarrow \mathcal{H}'$ such that

$$\begin{aligned} V\rho_g &= \rho'_g V \\ \Longrightarrow V^\dagger \rho'_g &= \rho_g V^\dagger \\ \Longrightarrow VV^\dagger \rho'_g &= V\rho_g V^\dagger = \rho'_g VV^\dagger. \end{aligned}$$

So each ρ_g commutes with $VV^\dagger$. Let $A = \sqrt{VV^\dagger}$, so ρ_g commutes with A. Therefore ρ_g commutes with $A^{-1}V$. But $A^{-1}V$ is unitary. □

Definition. The intertwining number of ρ, ρ' is

$$c(\rho, \rho') = \dim \mathrm{Hom}_G(\rho, \rho').$$

3.2. Theorem (Continuity of Unitary Representations). *Let ρ be a unitary representation of G on $\mathcal{H}$. Then T.F.A.E.*

1. *ρ is strongly continuous.*
2. *ρ is weakly continuous.*
3. *$g \mapsto (\rho_g u, u)$ is continuous at the identity for all $u \in \mathcal{H}$.*

Proof. 1⟹2⟹3 follows from the definitions. We will prove 3⟹1. Let $u \in \mathcal{H}$, $x, y \in G$.Then

$$\begin{aligned} \|\rho_x u - \rho_y u\| &= 2\,(u,u) - 2\mathrm{Re}\,(\rho_y u, \rho_x u) \\ &\le 2|\,(u,u) - (\rho_y u, \rho_x u)\,| \\ &\le 2|\,(u,u) - (\rho_{x^{-1}y} u, u)\,| \end{aligned}$$

The implication follows from this inequality. □

Definition. A representation is called (topologically) irreducible if it has no proper closed invariant subspaces.

Remark. For unitary representations there exists a notion of orthogonal complement for an invariant subspace. This allows a straightforward decomposition of representations.

3.3. Theorem (Orthogonal Complements). *Let ρ be a unitary representation of G on $\mathcal{H}$. Let $\mathcal{H}_1$ be a subspace of $\mathcal{H}$ with associated projection P. Then*

1. *$\mathcal{H}_1$ is invariant if and only if $P\rho_g = \rho_g P$, $g \in G$.*
2. *$\mathcal{H}_1^\perp$ is invariant if and only if $\mathcal{H}_1$ is invariant.*

Proof. This is a simple exercise. □

Definition. A representation ρ of G on $\mathcal{H}$ is called completely reducible if it is a direct sum of irreducible subrepresentations,

$$\mathcal{H} = \oplus_i \mathcal{H}_i, \quad P_i \rho_g = \rho_g P_i.$$

Remark. Here is a catch-all counter-example. Let $G = \mathbb{R}$. Let $\mathcal{H} = \mathbb{R}^2$. Let ρ be a representation in terms of upper-triangular matrices

$$\rho_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Clearly $\mathcal{H}_1 = \{(u, 0) \in \mathbb{R}^2\}$ is invariant. However, the complement $\mathcal{H}_1^\perp$ is not invariant. This shows that unitarity is necessary for the theorem on orthogonal complements.

Note further that $\mathcal{H}_1$ is a proper closed invariant subspace. However ρ is not completely reducible. The following theorem shows that unitarity cures this sickness as well, at least in the finite-dimensional case.

3.4. Theorem (Finite Unitary Reducibility). *Let ρ be a finite-dimensional unitary representation. Then ρ is completely reducible.*

Proof. Split out the invariant subspaces and their orthogonal complements. Proceed by induction. The induction terminates because $\mathcal{H}$ is finite-dimensional. □

Remark. The generic infinite-dimensional case requires the direct integral. Given this machinery, an appropriately similar result holds. This is the Gelfand-Raikov theorem. See 3.4.

3.5. Theorem (Schur A). *Let ρ, ρ' be unitary irreducible representations of G on $\mathcal{H}, \mathcal{H}'$. Suppose $V : \mathcal{H} \longrightarrow \mathcal{H}'$ is a bounded linear transformation such that*

$$V\rho_g = \rho'_g V, \quad g \in G.$$

Then either $\mathcal{H} \cong \mathcal{H}'$ or $V = 0$.

Proof. As in the proof of the unitary intertwiner proposition, $V^\dagger V$ commutes with ρ. Let $A = \int \lambda dE(\lambda)$ be a spectral representation for $A = V^\dagger V$. Then $\rho E(\lambda) = E(\lambda)\rho$. Therefore every closed subspace $\mathcal{H}_\lambda = E(\lambda)\mathcal{H}$ is invariant. But ρ is irreducible by hypothesis, therefore $\mathcal{H}_\lambda = \mathcal{H}$ or $\mathcal{H}_\lambda = \{0\}$. Therefore $A = \lambda 1$. Similarly $VV^\dagger = \lambda' 1$ and $\lambda V = \lambda' V$, so either $\lambda = \lambda'$ and $V \neq 0$, or $V = 0$. Setting $U = \lambda^{-1/2} V$ for the case $V \neq 0$, we have $U^\dagger U = UU^\dagger = 1$. Therefore $\mathcal{H} \cong \mathcal{H}'$ by U. □

3.6. Theorem (Schur B). *Let ρ be a unitary representation of G on $\mathcal{H}$. Then ρ is irreducible if and only if*

$$V\rho_g = \rho_g V, g \in G \Longleftrightarrow V = \lambda 1.$$

Proof. Suppose that all operators commuting with ρ are multiples of 1. Then if P is a projection operator commuting with ρ, $P = \lambda 1$, and then $\mathrm{Ran}(P) = \{0\}$. Therefore the only possible invariant subspaces are $\mathcal{H}, \{0\}$; therefore ρ is irreducible.

Conversely if ρ is irreducible and V is some operator commuting with ρ, then the self-adjoint operators $V_+ = 1/2(V + V^\dagger), V_- = 1/2(V - V^\dagger)$ commute with ρ. But $V_\pm$ must then be multiples of 1 by the proof of [Schur A]. Therefore V is a multiple of 1. □

Remark. In the finite-dimensional case we can make due without the assumption of unitarity, as shown by the following theorem.

3.7. Theorem. *Let ρ be an irreducible representation of G on $\mathcal{H}$, $\dim \mathcal{H} < \infty$. Then $V\rho = \rho V$ $\Longleftrightarrow V = \lambda 1$.*

Proof. Suppose $V\rho = \rho V$. Let $N = \ker(V)$. Then $\{0\} = V\rho \cdot \ker(V)$. Therefore $\rho\ker(V) \subset \ker(V)$, and $\ker(V)$ is invariant. By assumption $\ker(V) = \mathcal{H}$ or $\{0\}$. Suppose then that $\ker(V) = \{0\}$, otherwise $V = 0$. Let λ be an eigenvalue of V. Then $V - \lambda 1$ commutes with ρ by explicit computation and so it must too have $\ker(()\, V - \lambda 1) = \mathcal{H}$ or $\{0\}$. But it cannot be $\{0\}$ because $V - \lambda 1$ cannot have an inverse. Therefore $\ker(V - \lambda 1) = \mathcal{H}$, and $V = \lambda 1$. □

Definition. A representation ρ of G is called a factor representation when the center of $\mathrm{Hom}_G(\rho, \rho)$ consists purely of multiples of the identity, $\mathcal{Z}(\mathrm{Hom}_G(\rho, \rho)) = \{\lambda 1\}$.

Definition. A representation is called primary if it cannot be represented as a sum of disjoint representations; $\rho \neq \rho_1 \oplus \rho_2$. Such a representation might have a proper invariant subspace, but it may be impossible to split the space.

Definition. A factor representation is called $\mathrm{Type}\ I$ if it contains an irreducible subrepresentation. A group G is called $\mathrm{Type}\ I$ when all its factor representations are $\mathrm{Type}\ I$.

Remark. We note the following important facts. See [Mac89, p. 61].

- G compact $\Longrightarrow G$ is $\mathrm{Type}\ I$.
- G locally compact and abelian $\Longrightarrow G$ is $\mathrm{Type}\ I$.
- G is semi-simple Lie $\Longrightarrow G$ is $\mathrm{Type}\ I$.
- Let G be a countably infinite discrete group. Then G is $\mathrm{Type}\ I$ if and only there exists $N \subset G$, N abelian and normal, with G/N finite.
- G connected and nilpotent $\Longrightarrow G$ is $\mathrm{Type}\ I$.
- There exist connected Lie groups which are not $\mathrm{Type}\ I$.

Remark. Let F be the free group on two generators. Then the regular representation of F is *not* $\mathrm{Type}\ I$, and so F is not $\mathrm{Type}\ I$. Note that F is the fundamental group of the plane $\mathbb{R}^2$ with two punctures. This illustrates that it is possible to get in trouble very quickly. The representation theory of F is quite difficult.

3.2 Locally Compact Abelian Groups

3.8. Theorem (One-Dimensionality). *Let ρ be an irreducible unitary representation of a locally compact abelian group. Then ρ is one-dimensional.*

Proof. $\rho_x \rho_y = \rho_y \rho_x$. By [Schur B] then $\rho_x = c(x)1$, where $c(x)$ is a number depending on x. Therefore any $\mathcal{H}_1 \subset \mathcal{H}$ with $\dim \mathcal{H}_1 = 1$ is invariant. But ρ is irreducible so $\dim \mathcal{H} = 1$. □

Definition. A character χ of G is a one-dimensional continuous unitary representation of G,

$$\chi : G \longrightarrow \mathbb{C},$$
$$|\chi(g)| = 1,$$
$$\chi(g_1)\chi(g_2) = \chi(g_1 g_2).$$

Let $\widehat{G}$ denote the set of all characters of G. Clearly $\widehat{G}$ is an abelian group.

Remark. Examples:

- $G = \mathbb{R}^n \Longrightarrow \widehat{G} = \mathbb{R}^n$.
- $G = \mathrm{U}(1) \Longrightarrow \widehat{G} = \mathbb{Z}$.

Notice the result $G \cong \widehat{\widehat{G}}$, for these cases. This is a general result called Pontryagin duality.

3.9. Theorem (Stone-Naimark-Ambrose-Godement). *Let ρ be a continuous unitary representation of a locally compact abelian G on $\mathcal{H}$. Let $\widehat{G}$ be the character group of G. Then there exists a projection-valued measure $dE(\widehat{x})$ on $\widehat{G}$ such that*

$$\rho(x) = \int_{\widehat{G}} \widehat{x}(x)\, dE(\widehat{x}).$$

Proof. The function $x \mapsto (\rho_x u, u)$, $u \in \mathcal{H}$, is positive definite. Therefore there is a finite Borel measure $\mu_{u,u}$ on $\widehat{G}$ such that

$$(\rho_x u, u) = \int_{\widehat{G}} \widehat{x}(x)\, d\mu_{u,u}(\widehat{x}),$$

by Bochner's theorem. Use polar decomposition to write $(\rho_x u, v)$ in terms of $(\rho_x u', u')$ for some u'. So there is a unique complex measure $\mu_{u,v}$ on $\widehat{G}$ with

$$(\rho_x u, v) = \int_{\widehat{G}} \widehat{x}(x)\, d\mu_{u,v}(\widehat{x}).$$

Now $d\mu_{u,v}$ is a bounded linear functional. Therefore by the Riesz lemma, for any Borel set $\widehat{B} \subset \widehat{G}$ there is an operator $E(\widehat{B})$ on $\mathcal{H}$ with $\left(E(\widehat{B})u, v\right) = \mu_{u,v}(\widehat{B})$. This operator-valued measure gives a spectral measure, as can be checked. So

$$(\rho_x u, v) = \int_{\widehat{G}} \widehat{x}(x)\ \left(dE(\widehat{x})u, v\right).$$

□

Remark. The preceding theorem completely characterizes the unitary representations of locally compact abelian groups. However, we can give some more details. In the next section we will prove the Peter-Weyl theorem. As a corollary we will see that if G is compact then $\widehat{G}$ is countable. Now, the projection-valued measures on such a space are simple; each one is an assignment of a projection operator to each point of the discrete space. Therefore, up to equivalence, a representation is specified by giving the dimension of the range of each projection $P_{\widehat{x}}, \widehat{x} \in \widehat{G}$.

Remark. In the noncompact case the measures on $\widehat{G}$ are more diverse. Let ν be a measure on $\widehat{G}$. Define a representation of G on $L^2\left(\widehat{G};\nu\right)$ given by

$$\rho^\nu(g)f(\widehat{x}) = \widehat{x}(g)f(\widehat{x}).$$

The projection-valued measure associated to ρ^ν is

$$P^\nu, \quad P^\nu_E f(\widehat{x}) = 1_E \cdot f(\widehat{x}) \quad E \subset \widehat{G}.$$

Such projection-valued measures are equivalent if and only if their underlying measures are in the same measure class [i.e have the same sets of measure zero], $P^{\nu_1} = P^{\nu_2} \iff \nu_1 \cong \nu_2$. A complete set of invariants for a unitary representation of a locally compact abelian G is a sequence of mutually singular measure classes on $\widehat{G}$,

$$C_\infty, C_1, C_2, \ldots .$$

See [Mac89].

3.3 Compact Groups

3.10. Theorem (Unitarization). *Let ρ be a representation of a compact group G on $\mathcal{H}$. Then there exists an equivalent inner-product on $\mathcal{H}$ such that ρ is unitary with respect to this inner product.*

Proof. Define the new inner product by

$$(u,v)' = \int_G dg\ (\rho_g u, \rho_g v)\,.$$

Because of the averaging procedure it is clear that $(\cdot,\cdot)$ is preserved by ρ. So ρ_g are isometries with domain equal to the whole Hilbert space, and therefore they are unitary. Now

$$\|u\|'^2 \le \sup_{x\in G} \|\rho(x)\| \int_G dg\ (u,u) \le C' \|u\|^2\,,$$

and

$$\|u\|^2 \le \sup_{x\in G} C \|\rho(x)u\|^2 \le C'' \|u\|'^2\,.$$

So $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. □

Remark. This shows that if the group G is compact, then it is sufficient to consider unitary representations because any given representation can be unitarized by an averaging procedure.

3.11. Theorem (Peter-Weyl). *Let ρ be a unitary irreducible representation of a compact group G, on a Hilbert space $\mathcal{H}$. Then $\dim\mathcal{H} < \infty$. Furthermore*

$$\int_G dg\ (\xi_1, \rho(g)\eta_1)\,\overline{(\xi_2,\rho(g)\eta_2)} = \frac{(\xi_1,\xi_2)\,\overline{(\eta_1,\eta_2)}}{\dim\mathcal{H}}.$$

Finally, if $\rho_1 \not\equiv \rho_2$ are irreducible representations then

$$\int_G dg\ (\xi_1, \rho_1(g)\eta_1)\,\overline{(\xi_2,\rho_2(g)\eta_2)} = 0.$$

Proof. The map $\xi_1 \mapsto \int_G (\xi_1, \rho(g)\eta_1)\,\overline{(\xi_2, \rho(g)\eta_2)}$ is clearly a bounded linear functional on $\mathcal{H}$. By the Riesz lemma it is equal to $\xi_1 \mapsto (\xi_1, \zeta)$ for some $\zeta \in \mathcal{H}$, which depends continuously on ξ_2. Therefore there exists an operator A with $\zeta = A\xi_2$; A depends on η_1, η_2. Now $(\xi_1, \zeta) = (\xi_1, A\xi_2)$, so we compute

$$\begin{aligned}(\rho(g)\xi_1, A\xi_2) &= \int_G dg'\ (\rho(g)\xi_1, \rho(g')\eta_1)\,\overline{(\xi_2, \rho(g')\eta_2)} \\ &= \int_G dg'\ (\rho(g)\xi_1, \rho(g')\eta_1)\,\overline{(\rho(g)\rho(g^{-1})\xi_2, \rho(g')\eta_2)} \\ &= \int_G dg''\ (\xi_1, \rho(g'')\eta_1)\,\overline{(\rho(g^{-1})\xi_2, \rho(g'')\eta_2)} \\ &= \left(\xi_1, A\rho(g^{-1})\xi_2\right).\end{aligned}$$

So $(\xi_1, \rho(g^{-1})A\xi_2) = (\xi_1, A\rho(g^{-1})\xi_2)$. Therefore $A \in \mathrm{Hom}_G\,(\rho, \rho)$. But since ρ is irreducible, by Schur's lemma $A = \lambda 1$.

A similar argument for η_1, η_2 then shows that

$$\int_G (\xi_1, \rho(g)\eta_1)\,\overline{(\xi_2, \rho(g)\eta_2)} = C\,(\xi_1, \xi_2)\,(\eta_1, \eta_2)\,.$$

Now if $\xi_1 = \xi_2 = \eta_1 = \eta_2$ then

$$C = \frac{1}{\|\xi\|^4} \int_G dg\,|\,(\xi, \rho(g)\xi)\,|^2$$

So clearly $C > 0$.

Let $\{e_i\}$ be a collection of n orthonormal vectors in $\mathcal{H}$. Then

$$\begin{aligned}\sum_{i=1}^{n} |\,(\xi, \rho(g)e_i)\,|^2 &\leq \|\xi\|^2 \\ \Longrightarrow \sum_{i=1}^{n} \int_G dg\,|\,(\xi_1, \rho(g)e_i)\,|^2 &\leq \|\xi\|^2\,, \quad \int_G dg = 1 \\ \Longrightarrow \sum_{i=1}^{n} C\,\|\xi\|^2 &\leq \|\xi\|^2 \\ \Longrightarrow C &\leq \frac{1}{n}.\end{aligned}$$

If $\mathcal{H}$ were infinite-dimensional then taking $n \to \infty$ would give $C = 0$. But $C > 0$. Therefore $\dim \mathcal{H} < \infty$.

Now let $\{e_i\}$ be a basis for $\mathcal{H}$. Then the computation gives equality, $C = 1/n$.

Finally consider the case of two representations ρ, ρ' which are inequivalent. Since $A \in \mathrm{Hom}_G\,(\rho, \rho')$, $A = 0$. Therefore $C = 0$. □

Definition. Every group admits the so-called regular representations carried by $L^2\,(G; \mu)$. We have already seen the left regular representation. Define the right regular representation by

$$\rho^R(g)f(g') = f(g'g), \quad f \in L^2\,(G; \mu)\,.$$

In general ρ^R is highly reducible.

3.12. Theorem (Right Regular Completeness). *Each irreducible unitary representation ρ of a compact group G is equivalent to a subrepresentation of the right regular representation, ρ^R.*

Proof. Let $D_{jk}(g)$ be the matrix elements of the given representation ρ. Consider the set of functions $f_k = \sqrt{\dim \rho} D_{1k}$ in $L^2(G; \mu)$. The span of $\{f_k\}$ in $L^2(G; \mu)$ gives a subrepresentation of ρ^R,

$$\begin{aligned} \rho^R(g_0) f_k(g) &= D_{1k}(g g_0) \sqrt{\dim \rho} \\ &= D_{1j}(g) D_{jk}(g_0) \sqrt{\dim \rho} \\ &= D_{jk}(g_0) f_j(g). \end{aligned}$$

This representation is clearly equivalent to ρ. ☐

Definition. Let ρ be a finite-dimensional representation. The character of ρ is the function on G given by $\chi_\rho(g) = \mathrm{Tr}\,(\rho(g))$.

Remark. Characters have the obvious properties, stated in the following lemma. These properties make characters homomorphisms from the representation ring into the ring of functions on G which are constant on conjugacy classes.

3.13. Lemma.

$$\begin{aligned} \chi_{\rho_1 \oplus \rho_2} &= \chi_{\rho_1} + \chi_{\rho_2} \\ \chi_{\rho_1 \otimes \rho_2} &= \chi_{\rho_1} \chi_{\rho_2}. \end{aligned}$$

3.14. Theorem (Schur Criterion). *Let ρ be an irreducible unitary representation of a compact group G. Then one and only one of the following holds.*

1. $\chi_\rho \neq \overline{\chi_\rho}, \quad \int_G \chi_\rho(g^2) dg = 0, \quad \rho \ncong \overline{\rho}$.
2. $\chi_\rho = \overline{\chi_\rho}, \quad \int_G \chi_\rho(g^2) dg = +1, \quad \rho \cong \overline{\rho}, \rho$ *is real.*
3. $\chi_\rho = \overline{\chi_\rho}, \quad \int_G \chi_\rho(g^2) dg = -1, \quad \rho \cong \overline{\rho}, \rho$ *is not real.*

Proof. Let $\mathcal{H}$ be the representation space for ρ. Let $\mathcal{H}_S = \mathcal{H} \otimes_S \mathcal{H}$, $\mathcal{H}_A = \mathcal{H} \wedge \mathcal{H}$, and ρ_S, ρ_A acting respectively. Now

$$\begin{aligned} \mathrm{Tr}\,(\rho_S(g)) &= \sum_{i \leq j} \frac{1}{2} \left(e_i \otimes e_j + e_j \otimes e_i, \rho_S(g)(e_i \otimes e_j + e_j \otimes e_i) \right) \\ &= \sum_{i \leq j} \left(e_i, \rho(g) e_i \right) \left(e_j, \rho(g) e_j \right) + \left(e_j, \rho(g) e_i \right) \left(e_i, \rho(g) e_j \right), \end{aligned}$$

and

$$\mathrm{Tr}\,(\rho_A(g)) = \sum_{i < j} \left(e_i, \rho(g) e_i \right) \left(e_j, \rho(g) e_j \right) - \left(e_j, \rho(g) e_i \right) \left(e_i, \rho(g) e_j \right).$$

Therefore

$$\begin{aligned}
\mathrm{Tr}\left(\rho_S(g)\right) - \mathrm{Tr}\left(\rho_A(g)\right) &= 2\sum_i \left(e_i, \rho(g)e_i\right)\left(e_j, \rho(g)e_j\right) + 2\sum_{i<j}\left(e_j, \rho(g)e_i\right)\left(e_i, \rho(g)e_j\right) \\
&= 2\sum i \leq j \left(e_j, \rho(g)e_i\right)\left(e_i, \rho(g)e_j\right) \\
&= \sum_i \left(e_i, \rho(g)^2 e_i\right) \\
&= \mathrm{Tr}\left(\rho(g)^2\right).
\end{aligned}$$

So $\chi_\rho(g^2) = \chi_{\rho_S}(g) - \chi_{\rho_A}(g)$. Now it is a simple fact that $\int_G \chi_\rho(g)dg$ is equal to the multiplicity of the identity representation in ρ. Therefore

$$\begin{aligned}
\int_G \chi_\rho(g^2)dg &= \int_G [\chi_{\rho_S}(g) - \chi_{\rho_A}(g)]dg \\
&= \dim \mathrm{Fix}_G\left(\mathcal{H}_S\right) - \dim \mathrm{Fix}_G\left(\mathcal{H}_A\right).
\end{aligned}$$

Consider the action of G on $\mathcal{H} \otimes \mathcal{H}$. Since $\mathcal{H} \otimes \mathcal{H} = \mathrm{Hom}_G\left(\mathcal{H}, \mathcal{H}\right)$, and since ρ is irreducible, Schur's lemma implies

$$\dim \mathrm{Fix}_G\left(\mathcal{H} \otimes \mathcal{H}\right) = 0 \text{ or } 1.$$

Since $\mathcal{H} \otimes \mathcal{H} = \mathcal{H} \otimes_S \mathcal{H} + \mathcal{H} \otimes_A \mathcal{H}$,

$$\dim \mathrm{Fix}_G\left(\mathcal{H}_S\right) + \dim \mathrm{Fix}_G\left(\mathcal{H}_A\right) = 0 \text{ or } 1.$$

Therefore either

$$\begin{aligned}
\dim \mathrm{Fix}_G\left(\mathcal{H}_S\right) &= 1 \text{ and } \dim \mathrm{Fix}_G\left(\mathcal{H}_A\right) = 0 \\
\text{or } \dim \mathrm{Fix}_G\left(\mathcal{H}_S\right) &= 0 \text{ and } \dim \mathrm{Fix}_G\left(\mathcal{H}_A\right) = 1 \\
\text{or } \dim \mathrm{Fix}_G\left(\mathcal{H}_S\right) &= \dim \mathrm{Fix}_G\left(\mathcal{H}_A\right) = 0.
\end{aligned}$$

Suppose ρ is equivalent to a real representation. Therefore it is orthogonal. Therefore $\sum_i e_i \otimes e_i$ is invariant in $\mathcal{H}_S$, and we have case 2. Conversely suppose that $\mathcal{H}_S$ contains an invariant element. This element can be diagonalized over $\mathbb{C}$ to a multiple of the identity, so $\rho(G) \subset \mathrm{O}\left(n\right)$. In this case we have the contrapositive of 3. This completes the classification. □

Definition. Let $\widehat{G}$ denote the set of irreducible unitary non-equivalent representations of G.

3.15. Theorem (Peter-Weyl II). *The set of functions* $\left\{D^\rho_{ij}(g) = \left(\xi_i, \rho(g)\xi_j\right) \,:\, \rho \in \widehat{G}\right\}$ *is complete in* $L^2\left(G; dg\right)$.

Proof. Let $\mathcal{L}$ be the completion of the span of $\left\{D^\rho_{ij}(g) \,:\, \rho \in \widehat{G}\right\}$. Consider $\mathcal{L}^\perp$. If we could show that $\mathcal{L}^\perp$ contained a non-trivial finite-dimensional right-invariant subspace, then we could consider the right regular representation restricted to that subspace

$$\begin{aligned}
\rho^R(g)\psi_k(g_0) &= \psi_k(g_0 g) \\
&= D^{s'}_{kj}(g)\psi_j(g_0), \quad \text{some } s' \in \widehat{G}.
\end{aligned}$$

But then $\psi_k(g) = D^{s'}_{kj}(g)\psi_j(e)$, so that $\psi \in \mathcal{L}$. But $\psi \in \mathcal{L}^\perp$; therefore $\psi = 0$, but then the right-invariant subspace was actually trivial. So, supposing $\mathcal{L}^\perp$ is non-trivial, let us construct a non-trivial right-invariant subspace.

Pick a continuous symmetric function $w(x)$ in $\mathcal{L}^\perp$, and define an operator $A^\perp$,

$$(A^\perp\psi)(x) = \int_G w(xy^{-1})\psi(y)\,dy.$$

Recall that symmetric means $w(x) = \overline{w(x^{-1})}$. $A^\perp$ is self-adjoint and compact. Furthermore, $\mathrm{Ran}(A^\perp) \subset \mathcal{L}^\perp$, as the following computation shows.

$$\begin{aligned}
\int_G (A^\perp\psi)(x)\overline{D^s_{jk}(x)}\,dx &= \iint dx\,dy w(xy^{-1})\psi(y)\overline{D^s_{ij}(x)} \\
&= \iint dx'\,dy w(x')\psi(y)\overline{D^s_{ij}(x'y)} \\
&= \iint dx'\,dy w(x')\psi(y)\overline{D^s_{ik}(x')D^s_{kj}(y)} \\
&= \int dx' w(x')\overline{D^s_{ik}(x')}\int dy\psi(y)\overline{D^s_{kj}(y)} \\
&= 0.
\end{aligned}$$

The final equality holds since the first term vanishes, since $w \in \mathcal{L}^\perp$.

So $A^\perp$ projects onto $\mathcal{L}^\perp$. Therefore any eigenfunction of $A^\perp$ is in $\mathcal{L}^\perp$. Since $A^\perp$ is compact it has eigenspaces of finite multiplicity. Furthermore,

$$\int w(xyz^{-1})\psi(z)dz = \int w(xz^{-1})\psi(zy)dz,$$

so $A^\perp\rho^R = \rho^R A^\perp$. Therefore any eigenspaces of $A^\perp$ will be invariant under ρ^R. As per the previous comments,this completes the proof. □

Definition. Given $s \in \widehat{G}$ and $j \in \{1, \ldots, \dim s\}$, define the set of $\dim s$ functions

$$Y^s_{(j)k}(g) = \sqrt{\dim s} D^s_{jk}(g), \quad k = 1, \ldots, \dim s.$$

Also define

$$\tilde{Y}^s_{(j)k}(g) = \sqrt{\dim s}\overline{D^s_{kj}(g)}, \quad k = 1, \ldots, \dim s.$$

Let $\mathcal{H}^s_{(j)}$ be the closure of the span of $\{Y^s_{(j)k}\}$ and similarly $\tilde{\mathcal{H}}^s_{(j)}$; $\mathcal{H}^s_{(j)}$ and $\tilde{\mathcal{H}}^s_{(j)}$ carry representations $\rho^s_{(j)}$, $\tilde{\rho}^s_{(j)}$.

3.16. Theorem (Regular Reduction).

$$\rho^R = \bigoplus_{s\in\widehat{G}} \dim s\ \rho^s,$$

$$\rho^L = \bigoplus_{s\in\widehat{G}} \dim s\ \rho^s.$$

Proof. Consider $\mathcal{H}^s_{(j)}$. The $\tilde{\mathcal{H}}^s_{(j)}$ are analogous. First, $\mathcal{H}^s_{(j)} \perp \mathcal{H}^{s'}_{(j')}$ for $(s,j) \neq (s',j')$, so the decomposition is unique, in the form

$$\rho^R = \oplus_{s\in\widehat{G}, j\in\{1,\ldots,\dim s}\mathcal{H}^s_{(j)}.$$

But $\rho^s_{(j)}(x)Y^s_{(j)k}(x_0) = D^s_{lk}(x_0)Y^s_{(j)l}(x)$; therefore $\rho^s_{(j)} = \rho^s_{(j')}$. Therefore

$$\oplus_{j=1,\ldots,\dim s}\mathcal{H}^s_{(j)} \cong \dim s \cdot \mathcal{H}^s_{(1)} \cong \dim s\mathcal{H}^s.$$

□

Definition. Let $\rho, s \in \widehat{G}$. Define the following operators which act on the space $\mathcal{H}$ carrying the representation ρ.

$$P^s_{pq} = \dim s \int_G \overline{D^s_{pq}(g)}\rho(g)\, dg.$$

3.17. Lemma. *We have the following properties.*

1. $(P^s_{pq})^\dagger = P^s_{qp}$.
2. $P^s_{pq}P^{s'}_{p'q'} = \delta^{ss'}\delta_{qp'}P_{pq'}$.
3. $\rho(g)P^s_{pq} = D^s_{rp}(g)P^s_{rq}$.
4. $P^s_{pq}\rho(g) = D^s_{qr}(g)P^s_{pr}$.

Proof. These are simple calculations. □

Remark. Consider the example $G = \mathrm{SO}\,(3)$. $g = (\phi,\theta,\psi)$, $\phi \in [0,2\pi)$, $\theta \in [0,\pi)$, $\psi \in [0,2\pi)$. [Euler angles].

$$\begin{aligned}
\rho(g) &= \exp(-i\phi J_z)\exp(-i\theta J_y)\exp(-i\psi J_z)\\
dg &= \frac{\sin\theta}{8\pi^2}d\phi d\theta d\psi\\
D^J_{m,m}(g) &= \left(\frac{1+\cos\theta}{2}\right)^m P^{0,2m}_{j-m}(\cos\theta)\exp[-im(\phi+\psi)]\\
P^J_{mm} &= \frac{2J+1}{8\pi^2}\int \overline{D^J_{m,m}(\phi,\theta,\psi)}\,\rho(\phi,\theta,\psi)\sin\theta\, d\phi\, d\theta\, d\psi.
\end{aligned}$$

Remark. Suppose we have a factor representation. For compact G it will be of the form $n_s\rho_s$, where n_s is the multiplicity. Let $P^s_p = P^s_{pp}$, which is a projection operator. Let $\mathcal{H}^s_p = P^s_p\mathcal{H}$, where $\mathcal{H}$ is the s-pace for the factor representation, $\mathcal{H} = \rho^s\oplus\rho^s\oplus\cdots\oplus\rho^s$. The set $\left\{\langle s;p| = P^s_{pq}u \,:\, p = 1,\ldots,\dim s\right\}$ for q fixed and $u \in \mathcal{H}$ fixed, transforms as a set of basis vectors in an irreducible representation e-quivalent to s,

$$\rho^s(g)\,\langle s;p| = D^s_{rp}P^s_{rq}u = D^s_{rp}\,\langle s;r|\,.$$

So if we are given a factor representation in a particular form, then we can find the irreducible subspaces by the following procedure.

1. Find $\mathcal{H}^s_q = P^s_q \mathcal{H}$, q arbitrary but fixed.
2. Choose an orthonormal basis of $\mathcal{H}^s_q$, $\{u_i\}$.
3. For each u_i, an irreducible subspace is given by $\left\{\langle s;p| = P^s_{pq} u_i \,:\, p = 1,\ldots,\dim s\right\}$. In this way we find up to $\dim s$ irreducible subspaces for the representation s.

Remark. Let $s_1, s_2 \in \widehat{G}$, acting on spaces $\mathcal{H}_1, \mathcal{H}_2$. Pick a basis for each space,

$$\begin{gathered}\{\langle s_1 p_1|\} \subset \mathcal{H}_1, \quad p_1 = 1,\ldots,\dim s_1 \\ \{\langle s_2 p_2|\} \subset \mathcal{H}_2, \quad p_2 = 1,\ldots,\dim s_2.\end{gathered}$$

Consider the decomposition of the tensor product,

$$s_1 \otimes s_2 = \oplus_{s\in\widehat{G}} n_s S.$$

Suppose that $n_s = 0$ or 1 in the above. Then we can label the basis elements for $\mathcal{H}^s$ as $\{\langle sp|\}$, $p = 1,\ldots,\dim s$. A basis for $\mathcal{H}^{s_1} \otimes \mathcal{H}^{s_2}$ is given by

$$\{\langle s_1 p_1| \otimes \langle s_2 p_2| \quad p_1 = 1,\ldots,\dim s_1 \;\; p_2 = 1,\ldots,\dim s_2\}.$$

Apply the operators P^s_{pq} to write the basis vectors of $\mathcal{H}^s$ in terms of those for $\mathcal{H}^{s_1} \otimes \mathcal{H}^{s_2}$,

$$\begin{aligned}\langle sp| = \frac{1}{N} P^s_{pq} & \langle s_1 p'_1| \otimes \langle s_2 p'_2| \\ & p = 1,\ldots,\dim s \\ & q, p'_1, p'_2 \text{ fixed but arbitrary.} \\ & N = \text{ normalization constant .}\end{aligned}$$

Explicitly we have

$$\langle sp| = \frac{1}{N_{p'p'_1p'_2}} \dim s \int_G \overline{D^s_{pp'}(g)} \rho(g) \langle s_1 p'_1| \langle s_2 p'_2| \,,$$

and

$$N_{p'p'_1p'_2} = |sp'\rangle \left(\langle s_1 p'_1| \otimes \langle s_2 p'_2|\right).$$

It is a simple calculation to show that

$$\left(|s_1 p_1\rangle \otimes |s_2 p_2\rangle\right) \langle sp| = \frac{1}{N_{p'p'_1p'_2}} \dim s \int_G \overline{D^s_{pp'}(g)} D^{s_1}_{p_1p'_1}(g) D^{s_2}_{p_2p'_2}(g)\, dg.$$

In particular

$$N_{p'p'_1p'_2} = \left[\int_G \overline{D^s_{pp'}(g)} D^{s_1}_{p_1p'_1}(g) D^{s_2}_{p_2p'_2}(g)\, dg\right]^{1/2}.$$

In the case that the multiplicities in $s_1 \otimes s_2$ are greater than 1, we will have n_s sets of basis vectors, one for each of the u_i in steps 2 and 3 of the procedure given in the above remark. So we would have $\langle sp|_i$ and independent coupling coefficients $N^i_{p'p'_1p'_2}$.

3.4 Infinite-Dimensional Representations

Remark. When the group is no longer compact, infinite-dimensional representations are indispensable. It turns out that we must generalize the concept of direct sum to a direct integral of spaces in order to properly describe the decomposition of representations. Furthermore, the concept of character must be generalized, and this will require some generalized function theory.

Definition. Let (X, μ) be a Borel measure space. Let X index a set of Hilbert spaces $\{\mathcal{H}_x\}$. Define a μ-measurable field of Hilbert spaces to be $(\{\mathcal{H}_x\}_{x\in X}, \Gamma)$ where

1. Γ is a subspace of $\prod_x \mathcal{H}_x$.
2. For every $\gamma \in \Gamma$, $x \mapsto \|\gamma x\|_{\mathcal{H}_x}$ is measurable.
3. Let $\sigma \in \prod \mathcal{H}_x$. If $x \mapsto (\sigma(x), \gamma(x))$ is measurable for all $\gamma \in \Gamma$, then $\sigma \in \Gamma$.
4. There exists $\gamma_1, \gamma_2, \dots \in \Gamma$ such that, for every $x \in X$, the closure of the set $\{\gamma_n(x)\}$ is equal to $\mathcal{H}_x$.

Definition. When the above hold, elements of Γ are then called μ-measurable vector fields. Define $\gamma \in \Gamma$ to be square-integrable if

$$\int \|\gamma(x)\|^2_{\mathcal{H}_x} \, d\mu(x) < \infty.$$

Definition. The Hilbert space of square-integrable γ as given above, is called the direct integral of $\mathcal{H}_x$, denoted

$$\int^{\oplus} \mathcal{H}_x \, d\mu(x).$$

Definition. Let $\{A_x\}$ be a family of operators on the spaces $\{\mathcal{H}_x\}$. Suppose that the function $x \mapsto \|A_x\|_{\mathcal{H}_x}$ is bounded almost everywhere. Then the vector field $x \mapsto A_x\gamma(x)$, for every $\gamma \in \Gamma$, is measurable. The set $\{A_x\}$ then represents a bounded operator on $\int^{\oplus} \mathcal{H}_x \, d\mu(x)$, which is denoted

$$A = \int^{\oplus} A_x \, d\mu(x).$$

Such an operator is called decomposable.

Remark. There are some interesting algebraic structures associated to locally compact groups. At this point some operator-algebra theory is useful.

Definition. Let $M^1(G)$ be the algebra (under convolution) of bounded complex measures on G. Define μ^* by $d\mu^*(g) = \overline{d\mu(g^{-1})}$. Then $\|\mu^*\| = \|\mu\|$, and $M^1(G)$ is an involutive Banach algebra.

Definition. If ρ is a unitary representation of G and $\mu \in M^1(G)$, let

$$\rho(\mu) = \int_G \rho(g) \, d\mu(g).$$

We could call this the Fourier transform of the measure μ at the point ρ, $\hat{\mu}(\rho)$. However, we will not need this terminology in the following.

3.18. Lemma. *$\mu \mapsto \rho(\mu)$ is a representation of the involutive algebra $M^1(G)$ on $\mathcal{H}_\rho$.*

Proof. This is a simple computation. □

Remark. $L^1(G; dg)$ is embedded in $M^1(G)$ in an obvious way, and the above procedure provides a representation of $L^1(G; dg)$ as well.

Definition. Complete the algebra $L^1(G; dg)$ in the norm $\|f\|_1 = \sup_\rho \|\rho(f)\|$, where the supremum is over all *-representations of the algebra $L^1(G; dg)$. The resulting algebra is called the C^*-algebra of G, $C^*(G)$.

Definition. A *-representation is called non-degenerate if the closure of the set

$$\left\{\rho(a)\xi \,:\, a \in L^1(G; dg), \xi \in \mathcal{H}\right\}$$

is equal to $\mathcal{H}$.

3.19. Theorem. *A representation ρ of the algebra $L^1(G; dg)$, or of $C^*(G)$, is generated by a unitary representation of G if and only if ρ is non-degenerate. In that case, the representation of the group is uniquely determined.*

Proof. See [Kir76, p. 144] and [Dix69, 13.3.1+13.3.4]. □

3.20. Theorem (Gelfand-Raikov). *Every unitary representation ρ of a locally compact group G can be decomposed into a direct integral of irreducible representations.*

Proof. See [Kir76, p. 146]. □

Remark. We have the following special case which is some interest in its own right. [GGPS69, p. 23].

3.21. Theorem (Compact Operator Gelfand-Raikov). *Let $(\rho, \mathcal{H})$ be a unitary representation of a locally compact group G. Suppose that for all $f \in C_0^\infty(G)$ $\rho(f)$ is a compact operator. Then $\mathcal{H}$ splits as a discrete sum of irreducible unitary representations of finite multiplicity,*

$$\mathcal{H} = \bigoplus m_i \mathcal{H}_i.$$

Proof. First consider the set of $f \in C_0^\infty(G)$ which are symmetric, $f(g) = \bar{f}(g^{-1})$. For such f, $\rho(f)$ is self-adjoint. By assumption it is compact, so it has a countable discrete spectrum of finite multiplicity, except possibly for the spectral point zero. Therefore we have the spectral representation

$$\mathcal{H}' = \bigoplus_{k=0}^{\infty} \mathcal{H}'(f, k), \quad \dim \mathcal{H}' < \infty \text{ for } k \neq 0.$$

Here $\mathcal{H}'$ is any invariant subspace (including $\mathcal{H}$ itself). Let

$$\mathcal{H}^\infty = \bigcup_{f \in C_0^\infty(G),\, f \text{ symmetric},\, k \neq 0} \mathcal{H}(f, k).$$

Suppose that $\mathcal{H}^\infty$ is a proper subspace of $\mathcal{H}$. Then we have $v \neq 0$ in $\mathcal{H} - \mathcal{H}^\infty$ with $\rho(f)v = 0$ for all f. But this is not possible, by a simple calculation with approximate identities. So $\mathcal{H}^\infty = \mathcal{H}$.

So every invariant subspace of $\mathcal{H}$ has a nonempty intersection with some $\mathcal{H}(f,k)$, $k \neq 0$, and no $\mathcal{H}(f,k)$ is left out. Pick some $\mathcal{H}(f,k)$, $k \neq 0$. We know that there are invariant subspaces intersecting this subspace. Take the minimal nonempty invariant subspace from these, call it $\mathcal{H}_1$. If it were reducible then each of its components would intersect the chosen $\mathcal{H}(f,k)$, but this would contradict its minimality. Therefore $\mathcal{H}_1$ is irreducible.

Continue this procedure inductively, thus obtaining

$$\mathcal{H} = \bigoplus \mathcal{H}_k, \quad \mathcal{H}_k = \text{ invariant irreducible}.$$

Suppose that there was an $\mathcal{H}_k$ which was not of finite multiplicity in the above. Pick some $\rho(f)$ having an eigenvector in $\mathcal{H}_k$ with eigenvalue $\lambda \neq 0$. By assumption this eigenvalue would be repeated infinitely many times in the spectral decomposition of $\rho(f)$ on $\mathcal{H}$, which is not possible. Therefore each $\mathcal{H}_k$ is of finite multiplicity. □

Definition. A unitary representation ρ is called completely continuous if for all $f \in L^1(G; dg)$ $\rho(f)$ exists and is a compact operator. Some conditions for complete continuity are

- G semisimple Lie $\Longrightarrow$all irreducible unitary representations are CCR.
- G connected nilpotent Lie $\Longrightarrow$*ditto*.

I think that this is different from the following concept, but I am not sure.

Definition. Recall the notion of $\mathrm{Type}\ I$ representations. A locally compact group is called tame or $\mathrm{Type}\ I$ if all its irreducible unitary representations are $\mathrm{Type}\ I$. See p. 20.

Definition. Suppose that, given a representation ρ of G, we can find a subalgebra $\mathcal{D}_\rho(G) \subset M^1(G)$ satisfying

1. $\mathcal{D}_\rho(G)$ is invariant under right and left translations.
2. ρ generates a representation $\tilde{\rho}$ of $\mathcal{D}_\rho(G)$ and can be reconstructed uniquely from $\tilde{\rho}$.
3. $\{\rho(\mu) \,:\, \mu \in \mathcal{D}_\rho(G)\}$ are trace class and $\tilde{\rho}$ is continuous.

Then the character of ρ is defined to be the linear functional $\chi \in \mathcal{D}'_\rho(G)$ given by

$$\langle \chi, \mu \rangle = \mathrm{Tr}\left(\tilde{\rho}(\mu)\right).$$

3.22. Theorem. *Let G be a locally compact group. Then T.F.A.E.*

1. *For each irreducible representation ρ on a space $\mathcal{H}$, the character of ρ is defined on a dense subalgebra $\mathcal{D}_\rho(G) \subset C^*(G)$, for which $\tilde{\rho}(\mathcal{D}_\rho(G)) \neq 0$.*
2. *G is tame.*

Proof. See [Kir76, p. 162] [Dix69]. □

3.23. Theorem. *An irreducible representation ρ of a locally compact group G is defined up to equivalence by its character.*

Proof. $\{a \in C^*(G) : a^*a \in \mathcal{D}_\rho(G)\}$ is a pre-Hilbert space with $(a,b)_\chi = \langle \chi, b^*a \rangle$. Complete this to a Hilbert space $\mathcal{H}_\chi$. The map $\phi : a \mapsto \tilde{\rho}(a)$ is an isomorphism of $\mathcal{H}_\chi$ with the space of Hilbert-Schmidt operators on $\mathcal{H}_\rho$, $HS(\mathcal{H}_\chi)$. The actions of left and right translations by G on $\mathcal{H}_\chi$ are mapped into left and right multiplication by $\rho(g)$ in $\mathcal{H}_\rho$. Therefore the representation of G in $\mathcal{H}_\chi$ by left translations is equivalent to ρ. □

3.5 Abstract Plancherel Theorem

Remark. Recall that for compact G we had the fundamental Peter-Weyl theory which gave

$$L^2(G; dg) \cong \bigoplus_{i \in \widehat{G}} V_i \otimes V_i^* = \bigoplus_{i \in \widehat{G}} \operatorname{Hom}(V_i, V_i).$$

The explicit map is given by

$$\begin{aligned} L^2(G; dg) &\to \operatorname{Hom}(V_i, V_i) \\ f &\mapsto \int_G f(g)\rho_i(g)\, dg. \end{aligned}$$

This was fundamental because of the implication that understanding $\widehat{G}$ requires only understanding $L^2(G; dg)$. The non-compact case is similar but significantly more technical. It is zapped by some heavy C^*-algebra theory.

Definition. Let $\mathcal{A}$ be a separable C^*-algebra. Then $\mathcal{A}$ is called postliminal if it satisfies any of the following equivalent conditions.

1. If π is an irreducible representation of $\mathcal{A}$, then $\pi(\mathcal{A})$ contains the ideal of compact operators on $\mathcal{H}_\pi$.

2. If π_1, π_2 are irreducible representations of $\mathcal{A}$ with $\ker(\pi_1) = \ker(\pi_2)$, then $\pi_1 \cong \pi_2$.

3. Let $\mathcal{H}$ be a Hilbert space and $\widehat{\mathcal{A}}_{\mathcal{H}}$ be the set of irreducible representations of $\mathcal{A}$ on $\mathcal{H}$. Let $\cong$ be the relation of equivalence of representations. Then $\widehat{\mathcal{A}}_{\mathcal{H}}/\cong$ is a countably separated space.

See [Dix69, p. 99-101] [Con94, p. 460]. The equivalence of these is a theorem of J. Glimm.

Remark. Let $\mathcal{A}$ be a separable postliminal algebra. Let $\widehat{\mathcal{A}}_n$ be the set $\widehat{\mathcal{A}}_n \subset \widehat{\mathcal{A}}$ consisting of representations of dimension n, for $n = 1, 2, \ldots, \infty$. Then there exists a Borel field of Hilbert spaces $\zeta \mapsto \mathcal{H}(\zeta)$, on $\widehat{\mathcal{A}}$, with the property

$$\zeta \in \widehat{\mathcal{A}}_n \Longrightarrow \mathcal{H}(\zeta) = \mathcal{H}_n.$$

This is called the canonical field on $\widehat{\mathcal{A}}$. See [Dix69, p.174-175].

Remark. There is a bijection between $\widehat{C^*(G)}$ and $\widehat{G}$ for locally compact groups G. This endows $\widehat{G}$ with a topology. See [Dix69, p. 284-285, 353].

Remark. Let G be locally compact and $\mathrm{Type}\ I$. This is equivalent to $C^*(G)$ being separable and postliminal. Let $\zeta \mapsto \mathcal{H}(\zeta)$ be the canonical field of Hilbert spaces on $\widehat{C^*(G)}$, defined above. Suppose further that G is unimodular. Then we have the following result.

3.24. Theorem. *There exists a unique measure $\widehat{\mu}$ on $\widehat{G}$, called the Plancherel measure, with the following properties.*

1. $L^2(G; dg) \cong \int_{\widehat{G}}^{\oplus} \mathcal{H}(\zeta) \otimes \overline{\mathcal{H}}(\zeta)\, d\widehat{\mu}(\zeta)$.
2. $\rho^L = \int^{\oplus} (\zeta \otimes 1)\, d\widehat{\mu}(\zeta), \quad \rho^R = \int^{\oplus} (1 \otimes \overline{\zeta})\, d\widehat{\mu}(\zeta)$.
3. *If $f \in C^*(G)$ then $\zeta \mapsto \mathrm{Tr}\,(\zeta(f))$ is a lower semi-continuous function on $\widehat{G}$, and $f(1) = \delta_1(f) = \int_{\widehat{G}} \mathrm{Tr}\,(\zeta(f))\; d\widehat{\mu}(\zeta)$.*
4. *If $f \in L^1(G) \cap L^2(G)$, then $\int_G |f(g)|^2 dg = \int_{\widehat{G}} \mathrm{Tr}\,(\zeta(f)\zeta(f)^*)\, d\widehat{\mu}(\zeta)$.*

Proof. These results are stated in [Dix69, p. 367-369]. They can be generalized to the non-separable case. ☐

3.6 Induced Representations

Remark. Induction is a procedure for constructing representations of G given representations of a subgroup H. In many cases all irreducible unitary representations of G arise by induction from a one-dimensional representation of a certain subgroup.

Definition. Let G be a locally compact group and H a closed subgroup of G. Let ρ_0 be a representation of H on $\mathcal{H}$. Define a linear space

$$L\,(G, H; \rho_0) = \{F : G \longrightarrow H \ :\ F \text{ measurable }, F(hg) = \rho_0(g)F(g)\}\,.$$

Definition. Restrict attention to representations of the form

$$\rho_0(h) = \left[\frac{\Delta_H(h)}{\Delta_G(h)}\right]^{1/2} U(h),$$

where U is a unitary representation of H. Define a scalar product on $L\,(G, H; \rho_0)$ by

$$(F_1, F_2) = \int_G (F_1(g), F_2(g))_{\mathcal{H}}\, m(g) d\mu_R(g).$$

Then it is easy to see that the invariance of $(\cdot, \cdot)$ is equivalent to

$$\int_H m(hg) d\mu_R(h) = 1, \quad \text{independent of } g.$$

Definition. Complete $L\left(G, H; \rho_0\right)$ with $(\cdot, \cdot)$ to obtain a Hilbert space which we denote $L^2\left(G, H; \rho_0\right)$. Define a representation ρ of G on $L^2\left(G, H; \rho_0\right)$ by

$$(\rho(g)F)(g_1) = F(g_1 g).$$

This representation is said to be induced from H by U. We write $\rho = \operatorname{Ind}\left(G, H, U\right)$.

Remark. In words, we have introduced functions which transform in a particular way under the action of H on the left, and on these functions we have a representation of G acting on the right. The following gives a fundamental relation between induced representations and homogeneous spaces.

Definition. A unitary representation of a coset space $H\backslash G$ on a Hilbert space $\mathcal{H}$ is a pair (t, p) where t is a unitary representation of G on $\mathcal{H}$ and p is a *-representation of $C_0(H\backslash G)$ as an algebra with pointwise multiplication, and $t(g)p(f)t(g^{-1}) = p(r_g f)$, where $(r_g f)(x) = f(xg)$ is the usual right action.

3.25. Theorem. *A unitary representation of G is induced from $H \subset G$ if and only if it can be extended to a unitary representation of $H\backslash G$.*

Proof. See [Kir76, p. 192] [Mac89]. □

3.7 Trace Formula: Compact Domain

Remark. Let $(\rho, \mathcal{H})$ be a unitary representation of a locally compact group G. Suppose that $f \in C_0\left(G\right)$. Recall the definition of the smearing operation

$$\rho(f) = \int_G f(g)\rho(g)dg.$$

Note that this is precisely the Fourier transform $\hat{f}(\rho)$, but we will make no use of harmonic analysis in the following. The following results are from [GGPS69].

3.26. Theorem. *Let Γ be a discrete subgroup of a locally compact group G such that $\Gamma\backslash G$ is compact. Let $\rho = \operatorname{Ind}\left(G, \Gamma, \rho_0\right)$ be an induced representation of G, induced from Γ. If $f \in C_0\left(G\right)$ then $\rho(f)$ is a trace-class compact integral operator.*

Proof. Let $h \in L^2\left(G, \Gamma; \rho_0\right)$ be an element of the induced representation space, so that

$$h(\gamma g) = \rho_0(\gamma)h(g), \quad \gamma \in \Gamma, g \in G.$$

By definition we have $\rho(g_0)h(g) = h(gg_0)$ and so

$$\rho(f)h(g_1) = \int_G f(g)h(g_1 g)dg.$$

So we have

$$\begin{aligned}
\rho(f)h(g_1) &= \int_G f(g_1^{-1}g)h(g)dg \\
&= \int_F \left(\sum_{\gamma\in\Gamma} f(g_1^{-1}\gamma g)\rho_0(\gamma)h(g)\right) dg,
\end{aligned}$$

where F is a fundamental domain for Γ in G (not necessarily unique of course). So we have

$$\rho(f)h(g_1) = \int_F K(g_1, g)h(g)dg,$$

with kernel

$$K(g_1, g) = \sum_{\gamma\in\Gamma} f(g_1^{-1}\gamma g)\rho_0(\gamma).$$

Because $f \in C_0(G)$ and F is compact, only a finite number of terms in this sum are nonzero. Therefore the kernel is a continuous function, and so it is the kernel of a compact integral operator. $\rho(f)$ is clearly trace-class since the domain F is compact. □

3.27. Theorem. *Let G, Γ, ρ be as above. Then ρ splits into a discrete sum of a countable number of irreducible unitary representations, each of finite multiplicity.*

Proof. This follows from the special compact operator case of the Gelfand-Raikov theorem 3.21. □

3.28. Theorem. *Let Γ be a discrete subgroup of a locally compact group G such that $\Gamma\backslash G$ is compact. Further assume that G is tame (Type I). Let $\rho = \mathrm{Ind}\,(G, \Gamma, \rho_0)$ be an induced representation of G, induced from Γ. Let $\rho = \oplus m_\kappa \rho_\kappa$ be the decomposition of ρ guaranteed by the above theorem. Then, assuming that both sides exist,*

$$\mathrm{Tr}\,(\rho(f)) = \sum_\kappa m_\kappa \chi_\kappa(f),$$

where χ_κ is the character of ρ_κ. This can be written more explicitly as

$$\int_F \left(\sum_{\gamma\in\Gamma} f(g^{-1}\gamma g)\mathrm{tr}\,(\rho_0(\gamma))\right) dg = \sum_\kappa m_\kappa \int_G f(g)\chi_\kappa(g)dg.$$

Proof. Since G is tame, the required characters exist, and the r.h.s exists. The l.h.s exists by assumption. The rest follows from the definition of the character as a trace. □

3.29. Theorem. *Let G, Γ, ρ be as above. Assume further that G is compact. Then we have*

$$m_\kappa = \frac{1}{\mathrm{Order}(\Gamma)} \sum_{\gamma\in\Gamma} \bar{\chi_\kappa}(\gamma)\mathrm{tr}\,(\rho_0(\gamma)).$$

Proof. We have for compact G, $\int_G \bar{\chi_a}(g)\chi_b(G)dg = \delta_{ab}$. Insert this into the trace formula. We get

$$\int_F dg \left(\sum_{\gamma\in\Gamma} \chi_\kappa(\bar{g}^{-1}\gamma g)\mathrm{tr}\,(\rho_0(\gamma))\right) = m_\kappa$$

$$\int_F dg \sum_{\gamma\in\Gamma} \bar{\chi_\kappa}(\gamma)\mathrm{tr}\,(\rho_0(\gamma)) = m_\kappa,$$

and $\int_F dg = 1/\mathrm{Order}(\Gamma)$. □

3.8 Discrete Series

Definition. Let $(\rho, \mathcal{H})$ be a unitary and irreducible representation of a locally compact unimodular group G. ρ is said to be square-integrable if there exist $v, w \in \mathcal{H}$ such that

$$(v, \rho(g)w) \in L^2(G; dg).$$

Definition. The discrete series of G, denoted $\widehat{G}_{\text{disc}}$, is the set of square-integrable unitary irreducible representations of G.

3.30. Theorem. *Let G be a locally compact unimodular group and let $\rho \in \widehat{G}$. Then T.F.A.E.*

- $\rho \in \widehat{G}_{\text{disc}}$.
- $(v, \rho(g)w) \in L^2(G; dg)$ *for all* $v, w \in \mathcal{H}$.
- ρ *is a subrepresentation of the right regular representation.*

Proof. See [Rob83, p. 153]. □

Remark. Recall that semisimple Lie groups are unimodular.

Remark. If $\rho \in \widehat{G}_{\text{disc}}$ we write $L^2(G, \rho; dg)$ for the subrepresentation of the right regular representation equivalent to ρ.

3.31. Theorem (Formal Dimension). *Let* $(\rho, \mathcal{H}) \in \widehat{G}_{\text{disc}}$. *Then there is a constant* $d(\rho) \in \mathbb{R}, d(\rho) > 0$, *such that*

$$(\rho_{u_1,v_1}, \rho_{u_2,v_2}) = \frac{1}{d(\rho)} \overline{(u_1, u_2)}\, (v_1, v_2, .)$$

Proof. Define $\Theta(u) : \mathcal{H} \longrightarrow L^2(G; dg)$ by $\Theta(u)v = \rho_{uv}$. Then

$$\begin{aligned}(\rho_{u_1,v_1}, \rho_{u_2,v_2}) &= (\Theta(u_1)v_1, \Theta(u_2)v_2) \\ &= \left(v_1, \Theta^\dagger(u_1)\Theta(u_2)v_2\right).\end{aligned}$$

It is easy to see that $\Theta^\dagger(u_1)\Theta(u_2)$ commutes with ρ. Therefore it is a multiple of the identity, $c(u_1, u_2)1$. Similarly for v_1, v_2. So we have

$$\frac{\overline{(u_1, u_2)}}{c(u_1, u_2)} = \frac{\overline{(v_1, v_2)}}{c(v_1, v_2)} \equiv d(\rho).$$

Take $u_1 = u_2$, $v_1 = v_2$ and see that $d(\rho) > 0$. □

3.32. Theorem. *Let* $(\rho, \mathcal{H})$, $(\rho', \mathcal{H}') \in \widehat{G}_{\text{disc}}$. *If* $\rho \not\equiv \rho'$, *then* $(\rho_{uv}, \rho_{u'v'}) = 0$ *for all* $u, v \in \mathcal{H}$, $u', v' \in \mathcal{H}'$.

Proof.

$$\begin{aligned}|\,(\rho_{uv}, \rho'_{u'v'})\,| &\le \|\rho_{uv}\|\, \|\rho'_{u'v'}\| \\ &\le C(u, u')\, \|v\|\, \|v'\|,\end{aligned}$$

by the previous theorem. So there exists an operator $A(u, u') : \mathcal{H}' \longrightarrow \mathcal{H}$ with $(\rho_{uv}, \rho'_{u'v'}) = (v, A(u, u')v')$. By the Riesz lemma there is a $\bar{v} \in \mathcal{H}$ with $A(u, u')v' = \bar{v}$. Now $\|\bar{v}\|^2 \le C\, \|A(u, u')v'\|\, \|v'\|$, so $\|\bar{v}\| \le C'\, \|v'\|$. Therefore $A(u, u')$ is bounded. But it is easy to see that $A(u, u')$ intertwines ρ and ρ'. This contradicts the assumption that they are not equivalent. Therefore $A(u, u') = 0$. □

3.33. Theorem (Formal Dimension II). *Let* $\rho \in \widehat{G}_{\mathrm{disc}}$. *Suppose that* $\dim \rho < \infty$. *Then* G *is compact and* $d(\rho) = \dim \rho / \mu\,(G)$.

Proof. Let $\{e_i\}$ be an orthonormal basis of $\mathcal{H}$; $\{e_i\}$ is finite by assumption, and

$$\sum_i |\,(e_i, \rho(g)e_i)\,|^2 = 1$$

for any $g \in G$. By the previous theorem we have

$$\begin{aligned}
\frac{\dim \rho}{d(\rho)} &= \sum_i \frac{1}{d(\rho)} \overline{(e_i, e_i)}\,(e_j, e_j) \\
&= \sum_i \int_G dg |\,(e_i, \rho(g)e_j)\,|^2 \\
&= \int_G dg \sum_i |\,(e_i, \rho(g)e_j)\,|^2 \\
&= \mu\,(G)\,.
\end{aligned}$$

Therefore $\mu\,(G) < \infty$. □

Definition. If $L^2\,(G; dg) = \oplus_{\rho \in \widehat{G}_{\mathrm{disc}}} L^2\,(G, \rho; dg)$, then G is called a Fell group.

Remark. There exist non-compact examples of Fell groups. See [Rob83].

Chapter 4

Lie Algebras

4.1 Introduction

Definition. Define a Lie algebra to be a real vector space $\mathfrak{g}$ together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

1. $[x, y] = -[y, x]$.
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

Definition. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called an ideal if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$.

Definition. Let $\mathrm{ad}\,(X)$ be given by $\mathrm{ad}\,(X) : Y \mapsto [X, Y]$. Then $\mathrm{ad}\,(X)$ is an automorphism of $\mathfrak{g}$.

Definition. The Killing form of $\mathfrak{g}$ is the quadratic form $(X, Y) \equiv \mathrm{Tr}\,(\mathrm{ad}\,(X)\,\mathrm{ad}\,(Y))$.

Definition. $\mathfrak{g}$ is called semisimple if the Killing form $(\cdot,\cdot)$ is non-degenerate. $\mathfrak{g}$ is called simple if it is semisimple and has no non-trivial ideals.

Definition. A representation of a Lie algebra $\mathfrak{g}$ over a field K is a homomorphism $\mathfrak{g} \to \mathfrak{gl}_n\,(K)$, for some n.

Definition. Let $\mathfrak{g}$ be a Lie algebra. Note that $\mathrm{ad}\,(\cdot)$ provides a representation of $\mathfrak{g}$ on itself. If this adjoint representation decomposes as a direct sum of irreducible representations, then $\mathfrak{g}$ is called reductive.

4.1. Theorem. *A Lie algebra $\mathfrak{g}$ is reductive if and only if $\mathfrak{g} = \mathfrak{a} + \mathfrak{s}$, with $\mathfrak{a}$ abelian and $\mathfrak{s}$ semisimple. Furthermore, when this holds we have $\mathfrak{a} = \mathcal{Z}\,(\mathfrak{g})$, the center of $\mathfrak{g}$, and $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$.*

Proof. By assumption $\mathfrak{g}$ splits as $\oplus \mathfrak{g}_i$, where $\mathrm{ad}\,(\cdot)$ acts irreducibly on each $\mathfrak{g}_i$. Each $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$ since $\mathrm{ad}\,(\cdot)$ acts irreducibly. Therefore each $\mathfrak{g}_i$ is either simple or one-dimensional. Take $\mathfrak{s}$ to be the sum of the simple $\mathfrak{g}_i$ and $\mathfrak{a}$ to be the sum of the one-dimensional $\mathfrak{g}_i$. $\square$

4.2 $\mathfrak{sl}_2(\mathbb{C})$

Remark. The simplest non-trivial Lie algebra is the algebra of 2×2 complex matrices with vanishing trace, $\mathfrak{sl}_2(\mathbb{C})$. The structure of this algebra is very important for the general theory, and so it is convenient to study it first.

4.2. Lemma. *The following elements provide a basis for $\mathfrak{sl}_2(\mathbb{C})$ over $\mathbb{C}$.*

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, they satisfy the relations

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Proof. This is a trivial observation. □

4.3. Theorem. *Let $m \geq 1$ be an integer. Then up to equivalence there exists a unique irreducible complex-linear representation of dimension m, π, of $\mathfrak{sl}_2(\mathbb{C})$. The representation space has a basis $\{v_0, \ldots, v_{m-1}\}$ such that*

- $\pi(h)v_i = (m - 1 - 2i)v_i$
- $\pi(e)v_0 = 0$
- $\pi(f)v_i = v_{i+1}$
- $\pi(e)v_i = i(m - i)v_{i-1}$

Proof. □

Definition. The one-dimensional subalgebra generated by h is called the Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$.

Definition. Let π be the unique irreducible representation of dimension m guaranteed by the theorem. The set of eigenvalues of $\pi(h)$, $\{m-1, m-3, \ldots, -(m-3), -(m-1)\}$, is called the set of weights for the m-dimensional representation. The vector v_0 is called the highest weight vector.

4.3 Weights and Roots

Remark. The infinitesimal theory is probably the most important tool for Lie groups. We will be able to construct representations of the group from representations of the associated Lie algebra. In the compact case, the classical theory of Lie algebras is powerful enough to yield all the unitary representations of semisimple Lie groups. The representations will be constructed from certain $\mathfrak{g}$-modules.

Remark. Recall that $\mathfrak{g}$ was defined to be semisimple if the Killing form was non-degenerate. This is actually a somewhat involved idea. From the algebraic standpoint we might expect semisimplicity to be a property depending on the structure of ideals of $\mathfrak{g}$.

Definition. The derived series of $\mathfrak{g}$ is the decreasing sequence of ideals defined by

$$\mathfrak{D}^{r+1} = [\mathfrak{D}^r, \mathfrak{D}^r], \quad \mathfrak{D}^0 = \mathfrak{g}.$$

Definition. $\mathfrak{g}$ is said to be solvable if $\mathfrak{D}^k = 0$ for some k.

Definition. The radical of $\mathfrak{g}$, $\mathrm{rad}\,(\mathfrak{g})$, is the unique maximal solvable ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is finite-dimensional, the radical exists.

Definition. $\mathfrak{g}$ is said to be semisimple when $\mathrm{rad}\,(\mathfrak{g})$ = 0. One of our goals will be to prove the equivalence of this definition to the previous definition in terms of the Killing form.

Definition. An element $X \in \mathfrak{g}$ is called a semisimple element if $\mathrm{ad}\,(X) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is diagonalizable over $\mathbb{C}$.

Definition. The Cartan subalgebra of $\mathfrak{g}$ is the maximal abelian subalgebra of semisimple elements of $\mathfrak{g}$. Equivalently, it is a subalgebra which is nilpotent and which is its own normalizer in $\mathfrak{g}$. [Jac62, p. 57].

Definition. Let ρ be a representation of $\mathfrak{g}$ on a space V. A function $\alpha : \mathfrak{g} \longrightarrow \mathbb{C}$ is called a weight of V if there is a nonzero vector $v \in V$ such that

$$(\rho(x) - \alpha(x)1)^k \, v = 0, \quad \text{for some } k.$$

The set of such vectors together with $0 \in V$ is a subspace of V called the weight space of α, V_α.

Definition. Let ρ^* be the representation of $\mathfrak{g}$ acting on V^* by

$$(x, \rho^*(X)y) = -\left(\rho(X)x, y\right), \quad x \in V, y \in V^*, X \in \mathfrak{g}.$$

Notice that $(\lambda - \alpha(X))^{\dim V}$ is the characteristic polynomial for $\rho(X)$, so $(\rho(X) - \alpha(X)1)^{\dim V} v = 0$ for all $v \in V$.

4.4. Lemma. *Let V be the weight space for α. Then V^* is the weight space for $-\alpha$.*

Proof.

$$\begin{aligned}
&(x, \rho^*(X)y) = -\left(\rho X x, y\right) \\
&\Longrightarrow (x, \rho^*(X)y) + (x, \alpha(X)y) = -\left(\rho(X)x, y\right) + (\alpha(X)x, y) \\
&\Longrightarrow (x, (\rho^*(X) + \alpha(X)1)y) = -\left((\rho(X) - \alpha(X)1)x, y\right).
\end{aligned}$$

By iteration then

$$\left(x, (\rho^*(X) + \alpha(X)1)^k y\right) = -\left((\rho(X) - \alpha(X)1)^k x, y\right).$$

Let $k = \dim V = \dim V^*$. Then the right-hand side vanishes for all x since V is a weight space for α. Thus $(\rho^*(X) = \alpha(X)1)^k y = 0$ for all $y \in V^*$. So V^* is a weight space for $-\alpha$. ☐

4.5. Lemma. *Let V_1 and V_2 be weight spaces for α and β respectively. Then $V_1 \otimes V_2$ is a weight space for $\alpha + \beta$.*

Proof. The tensor product representation $\rho = \rho_1 \otimes \rho_2$ acts by $\rho(x_1 \otimes x_2) = \rho_1(x_1) \otimes x_2 + x_1 \otimes \rho_2(x_2)$. A simple computation gives

$$[\rho(X) - (\alpha(X) + \beta(X))]^m (x_1 \otimes x_2) = \sum_{i=0}^{m} [\rho_1(X) - \alpha(X)]^i \cdot x_1 \otimes [\rho_2(X) - \beta(X)]^{m-i} \cdot x_2.$$

Take $m = k + k' - 1$ where k and k' define the weight spaces V_1, V_2. Then all the terms on the right-hand side vanish, so $[\rho(X) - (\alpha(X) + \beta(X))]^m (x_1 \otimes x_2) = 0$. □

Definition. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then the representation ρ of $\mathfrak{g}$ on V also gives a representation of $\mathfrak{h}$. Of particular interest is the case where $\rho = \mathrm{ad}\,()$, the adjoint representation. The weights associated to the adjoint representation are called the roots of $\mathfrak{h}$ in $\mathfrak{g}$.

Remark. Suppose that $\mathrm{ad}\,(Y)$ is diagonalizable for all $Y \in \mathfrak{h}$. This will occur when the eigenvalues of the matrices $\mathrm{ad}\,(Y)$ lie in the base field; in particular it is automatic if the base field is algebraically closed. Then the algebra $\mathfrak{g}$ will split as a sum of root spaces

$$\mathfrak{g} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \cdots \oplus \mathfrak{g}_{\alpha_n}.$$

4.6. Theorem. *If $\alpha + \beta$ is a root then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. Otherwise $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.*

Proof. The set $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ is the image of the linear space $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta$ under a map π defined by

$$\pi(\sum_i X_\alpha^{(i)} \otimes X_\beta^{(i)}) = \sum_i \mathrm{ad}\left(X_\beta^{(i)}\right) \cdot X_\alpha^{(i)}.$$

Now we can calculate

$$\begin{aligned} \mathrm{ad}\,(Y) \cdot [\pi(X_\alpha \otimes X_\beta)] &= \mathrm{ad}\,(Y) \cdot [\mathrm{ad}\,(X_\beta) \cdot X_\alpha], \quad Y \in \mathfrak{h} \\ &= \mathrm{ad}\,(X_\beta) \cdot [\mathrm{ad}\,(Y) \cdot X_\alpha] + \mathrm{ad}\,([X_\beta, Y]) \cdot X_\alpha \\ &= \pi((\mathrm{ad}\,(Y) \cdot X_\alpha) \otimes X_\beta + X_\alpha \otimes [X_\beta, Y]) \\ &= \pi(\mathrm{ad}\,(Y) \cdot (X_\alpha \otimes X_\beta)). \end{aligned}$$

So π is a homomorphism of $\mathrm{ad}\,(\mathfrak{h})$ modules. Now apply the previous result regarding $V_1 \otimes V_2$. □

Remark. Let $\mathfrak{h}$ be a Cartan subalgebra. Then $\mathfrak{h}$ is the root space for the trivial root $\alpha = 0$. Recall $\mathfrak{D}^1 = [\mathfrak{g}, \mathfrak{g}]$; note that $\mathfrak{g}$ is solvable if and only if $\mathfrak{D}^1$ is nilpotent.

Remark. Let $R_\pm$ be the set of roots α such that $-\alpha$ is also a root, so that the following makes sense. Then we have

$$\mathfrak{h} \bigcap \mathfrak{D}^1 = \sum_{\alpha \in R_\pm} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}].$$

This is a simple consequence of the definitions since

$$\mathfrak{D}^1 = \sum_{\alpha,\beta} [\mathfrak{g}_\alpha, \mathfrak{g}_\beta], \quad \text{and } \mathfrak{h} = \mathfrak{g}_0.$$

4.4 Cartan Theorems

4.7. Lemma. *Let the base field be algebraically closed and of characteristic zero. Let ρ be a representation of $\mathfrak{g}$ on V. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\alpha \in R_{\pm}$, $e_\alpha \in \mathfrak{g}_\alpha$, $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$, and $\mathfrak{h}_\alpha = [e_\alpha, e_{-\alpha}]$. If β is a weight of $\mathfrak{h}$ in V,then $\beta(h_\alpha) = q\alpha(h_\alpha)$ for some $q \in \mathbb{Q}$.*

Proof. See [Jac62, p. 67]. ☐

4.8. Theorem (Cartan). *Let the base field be of characteristic zero. Let ρ be a representation of $\mathfrak{g}$ on V, $\dim V < \infty$. Suppose*

1. $\ker(\rho)$ *is solvable in* $\mathfrak{g}$.
2. $\mathrm{Tr}\,(\rho(X)^2)$ = *0 for all* $X \in \mathfrak{D}^1 = [\mathfrak{g}, \mathfrak{g}]$.

Then $\mathfrak{g}$ is solvable.

Proof. Assume $\mathfrak{D}^1 = \mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We have the decomposition into weight spaces

$$V = V_{\beta_a} \oplus V_{\beta_2} \oplus \cdots \oplus V_{\beta_k},$$

and the decomposition of the adjoint representation space

$$\mathfrak{g} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \cdots \oplus \mathfrak{g}_{\alpha_k}.$$

$\mathfrak{h} = \mathfrak{h} \cap \mathfrak{D}^1$, so $\mathfrak{h} = \sum_{\alpha \in R_\pm} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Let $e_\alpha \in \mathfrak{g}_\alpha$, $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$, and let $h_\alpha = [e_\alpha, e_{-\alpha}]$.
Now $\rho(h_\alpha)$ restricted to some V_β has a single characteristic root $\beta(h_\alpha)$, since V_β is a weight space. Therefore $\rho(h_\alpha)^2$ restricted to V_β has a single characteristic root $\beta(h_\alpha)^2$. By assumption $0 = \mathrm{Tr}\,(\rho(h_\alpha)^2) = \sum_\beta \dim V_\beta \beta(h_\alpha)^2$. By the previous lemma $\beta(h_\alpha) = r_\beta \alpha(h_\alpha)$ so $0 = \alpha(h_\alpha)^2 \sum_\beta r_\beta^2 \dim V_\beta$, and so $\alpha(h_\alpha) = 0$. Therefore $\beta(h_\alpha) = 0$, and this holds for any α. Therefore $\beta(\mathfrak{h}) = 0$, and so $V = V_0$, i.e. the only weight for V is $\beta = 0$. But then $\rho(\mathfrak{g}_\alpha)V = 0$ if $\alpha \neq 0$.

$$\begin{aligned}
&\Longrightarrow \cup\{\mathfrak{g}_\alpha : \alpha \neq 0\} = \mathfrak{g} \ominus \mathfrak{h} \subseteq \ker(\rho) \\
&\Longrightarrow \ker(\rho) = \mathfrak{g} \ominus \mathfrak{h} \oplus \mathfrak{k} = (\mathfrak{g} \oplus \mathfrak{k}) \ominus \mathfrak{h} \\
&\Longrightarrow \mathfrak{g}/\ker(\rho) \cong \mathfrak{h}/\mathfrak{k} \\
&\Longrightarrow \mathfrak{g}/\ker(\rho) \text{ is nilpotent, since } \mathfrak{h} \text{ is nilpotent.}
\end{aligned}$$

By assumption $\ker(\rho)$ is solvable, so then $\mathfrak{g}$ is solvable. But we assumed $\mathfrak{D}^1 = \mathfrak{g}$, in which case $\mathfrak{D}^1 = \mathfrak{D}^2 = \mathfrak{D}^k = \cdots = \mathfrak{g}$ never terminates, and thus $\mathfrak{g}$ is not solvable. $\Rightarrow\Leftarrow$. Therefore $\mathfrak{D}^1$ is properly contained in $\mathfrak{g}$, $\mathfrak{D}^1 \subset \mathfrak{g}$.

Now, if $\mathfrak{g}$ satisfies both conditions, then so does $\mathfrak{D}^1$. Therefore we can repeat the argument for $\mathfrak{D}^1$. Proceeding inductively we get a proper tower $\mathfrak{D}^k \subset \cdots \subset \mathfrak{D}^1 \subset \mathfrak{g}$. Since it is proper and $\mathfrak{g}$ is finite dimensional, the tower terminates, and $\mathfrak{g}$ is therefore solvable.

We assumed that the base field was algebraically closed, in order to apply the lemma. Now assume this is not the case. Let Ω be the algebraic closure of the field. Extend everywhere, so that V becomes a space over Ω, etc.

Solvability of $\ker(\rho)$ implies solvability of $\ker(\rho)_\Omega$. So the first condition holds for $\mathfrak{g}_\Omega$. Compute

$$\operatorname{Tr}(\rho(X)\rho(Y)) = \frac{1}{2}\left[\operatorname{Tr}\left((\rho(X)+\rho(Y))^2\right) - \operatorname{Tr}\left(\rho(X)^2\right) - \operatorname{Tr}\left(\rho(Y)^2\right)\right].$$

Now an element of $\mathfrak{g}_\Omega$ can be written $X_\Omega \in \mathfrak{g}_\Omega \equiv \mathfrak{g} \otimes_F \Omega = \sum_i \omega_i X_i$, $X_i \in \mathfrak{g}$. Therefore

$$\begin{aligned}\operatorname{Tr}\left(\rho(X_\Omega)^2\right) &= \sum_{i,j} \omega_i\omega_j \operatorname{Tr}(\rho(X_i)\rho(X_j)) \\ &= 0,\end{aligned}$$

using the above computation and the second condition for $\mathfrak{g}$. Therefore the second condition holds for $\mathfrak{g}_\Omega$ as well.

So the argument follows for $\mathfrak{g}_\Omega$, and therefore $\mathfrak{g}_\Omega$ is solvable. But this is equivalent to solvability of $\mathfrak{g}$. ☐

4.9. Theorem (Cartan II). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field F of characteristic zero. Then $\mathfrak{g}$ is semisimple if and only if the Killing form is non-degenerate.*

Proof. Let $(\cdot,\cdot)$ be the Killing form. Let $\mathfrak{g}^\perp = \{X : (X,Z) = 0, \forall Z \in \mathfrak{g}\}$; then $\mathfrak{g}^\perp$ is an ideal. Clearly $(X,X) = \operatorname{Tr}(\operatorname{ad}(X)\operatorname{ad}(X)) = 0$ for all $X \in \mathfrak{g}^\perp$. Therefore, by the previous theorem $\mathfrak{g}^\perp$ is solvable. But by assumption $\mathfrak{g}$ is semisimple, so $\mathfrak{g}^\perp = 0$, and so $(\cdot,\cdot)$ is non-degenerate.

Conversely, suppose $\mathfrak{g}$ is not semisimple. Then there exists an abelian ideal $\mathfrak{k} \subset \mathfrak{g}$, $\mathfrak{k} \neq 0$. But then clearly $\mathfrak{k} \subseteq \mathfrak{g}^\perp$, and so $(\cdot,\cdot)$ is degenerate. ☐

Remark. These proofs are taken from Jacobson [Jac62], with some changes of notation and a few missing steps inserted.

Remark. The following shows that this simple criterion has an important consequence which allows us to deal with non-algebraically closed fields.

4.10. Corollary. *Let $\mathfrak{g}_F$ be a Lie Algebra over a base field F. Then $\mathfrak{g}_F$ is semisimple if and only if $\mathfrak{g}_\Omega$ is semisimple for every extension field $\Omega \supset F$.*

Proof. The statement in terms of non-degeneracy of $(\cdot,\cdot)$ makes this result trivial. ☐

4.11. Theorem (Structure of Semisimple Algebras). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic zero. Then $\mathfrak{g}$ is semisimple if and only if*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r,$$

where $\mathfrak{g}_i$ are simple ideals.

Proof. See [Jac62, p. 71]. ☐

Remark. This result is often used as the definition of semisimplicity.

Remark. Summarizing, we have found decompositions of $\mathfrak{g}$, relative to any Cartan subalgebra $\mathfrak{h}$. It may be necessary to pass to an extension field in order to obtain such a decomposition.Typically this means complexifying an algebra $\mathfrak{g} \to \mathfrak{g}_\mathbb{C}$. Symbolically, the features are

$$\begin{gathered}[\mathfrak{h},\mathfrak{h}] = 0 \\ \mathfrak{g} = \mathfrak{g}_{\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{\alpha_k} \\ \operatorname{ad}(\mathfrak{h}) \cdot X_\alpha = \alpha(\mathfrak{h}) X_\alpha, \quad X_\alpha \in \mathfrak{g}_\alpha.\end{gathered}$$

Such a decomposition depends on the choice of $\mathfrak{h}$.

4.5 Root Systems

Definition. Let $\mathfrak{g}$ be a semisimple Lie algebra and pick a Cartan subalgebra $\mathfrak{h}$. Let R be the set of roots associated with $\mathfrak{h}$. R is called the root system for $\mathfrak{h}$. The span of R in $\mathfrak{h}^*$ is called the root space for $\mathfrak{h}$.

Remark. Since $\mathfrak{g}$ is semisimple, $(\cdot,\cdot)$ is non-degenerate. Then $X \mapsto (X,\cdot)$ defines an isomorphism of $\mathfrak{g}$ and $\mathfrak{h}^*$, and $(\cdot,\cdot)$ defines a bilinear form on $\mathfrak{h}^*$,

$$\langle X^*, Y^* \rangle \equiv (X, Y)\,.$$

The calculation in the Cartan theorem, showing that $\mathrm{Tr}\,(\rho(h_\alpha)^2) = \sum_\beta \dim V_\beta \beta(h_\alpha)^2$, shows that $\langle\cdot,\cdot\rangle$ is real and positive definite on the set $R \in \mathfrak{h}^*$.

Definition. The set of coroots is defined as

$$R' = \left\{ \frac{2\,\langle \alpha, \cdot \rangle}{\langle \alpha, \alpha \rangle} \,:\, \alpha \in R \right\} \subset (\mathfrak{h}^*)^* \cong \mathfrak{h}.$$

We can write $\alpha^V = 2\alpha / \langle \alpha, \alpha \rangle$,

$$\alpha^V \leftrightarrow \alpha$$
$$(\mathfrak{h}^*)^* \cong \mathfrak{h}$$

4.12. Theorem (Root Systems). *Let R be a root system. Then*

1. $\left\langle \alpha^V, \beta \right\rangle = \frac{2\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$, *for* $\alpha, \beta \in R$.
2. *If* $\alpha = c\beta$, $\alpha, \beta \in R$, *then* $c = -1, 0, +1$.

Proof. Pick any root α and consider the spaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$. Let $X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ and consider the subalgebra spanned by $\{X_\alpha, X{-}\alpha, [X_\alpha, X_{-\alpha}]\}$. The multiplication table is

$$[X_\alpha, X_{-\alpha}] = H_\alpha \equiv \frac{2\,\langle \alpha, \cdot \rangle}{\langle \alpha, \alpha \rangle}$$
$$[X_\alpha, H_\alpha] = -2X_\alpha$$
$$[X_{-\alpha}, H_\alpha] = 2X_{-\alpha}.$$

So this subalgebra is isomorphic to $\mathfrak{sl}_2\,(\mathbb{C})$. But by explicit construction (raising and lowering operators), the eigenvalues of $\rho(H_\alpha)$ are integer valued for any representation ρ.

Consider the representation $\rho(\cdot) = \mathrm{ad}\,(\cdot)$. So the eigenvalues of $\mathrm{ad}\,(H_\alpha)$ are integers. By definition of a root, if $\beta \in R$ then $\beta(H_\alpha)$ is an eigenvalue of $\mathrm{ad}\,(H_\alpha)$, so $\beta H_\alpha \in \mathbb{Z}$. But $\beta(H_\alpha) = 2\,\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle$. So we have the first property.

Now $\mathrm{Tr}\,(\mathrm{ad}\,(H_\alpha)) = -2 + 2 = 0$, but $\mathrm{Tr}\,(\mathrm{ad}\,(H_\alpha)) = (\alpha, \alpha)\,(1 - \dim \mathfrak{g}_{-\alpha} - 2\dim \mathfrak{g}_{-2\alpha} - 3\dim \mathfrak{g}_{-3\alpha} - \cdots)$. Since $(\alpha, \alpha) \neq 0$, the only solution is $\dim \mathfrak{g}_{-\alpha}$ = 1 and the rest vanishing. Therefore $-2\alpha, -3\alpha, \cdots \notin R$, and similarly for $+2\alpha, +3\alpha, \ldots$. $\square$

Remark. This result constrains the geometry of root systems severely. Given any two roots α_i, α_j, the angle between them satisfies

$$\begin{aligned}\cos^2\theta_{ij} = \frac{\langle\alpha_i,\alpha_j\rangle^2}{\langle\alpha_i,\alpha_i\rangle\langle\alpha_j,\alpha_j\rangle} &= \frac{1}{4}\left[\frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle}\right]\left[\frac{2\langle\alpha_j,\alpha_i\rangle}{\langle\alpha_j,\alpha_j\rangle}\right]\\ &= \frac{1}{4}mn, \quad m,n\in\mathbb{Z}\\ &\equiv \frac{1}{4}m_{ij}.\end{aligned}$$

Definition. Each root α defines a hyperplane normal to itself. Define reflection transformations about these hyperplanes by

$$w_\alpha(x) = x - \frac{2\langle\alpha,x\rangle}{\langle\alpha,\alpha\rangle}\alpha.$$

The set of all hyperplanes cuts the root space into a set of congruent simplicial cones called Weyl chambers. The finite group generated by $\{w_\alpha\}$ is called the Weyl group, $\mathrm{Weyl}\,(G)$ or $\mathrm{Weyl}\,(\mathfrak{g},\mathfrak{h})$. It permutes the elements of the set of roots.

Definition. Let R be a root system. Pick a Weyl chamber C. From each pair of roots $\pm\alpha\in R$, pick the one which has the property $\langle\alpha,\lambda\rangle > 0$ for all $\lambda\in C$. This set of roots with positive projection onto C is called the set of simple roots, $S(R)$.

Definition. Let R be a root system. A positive root system for R , $R_+\subset R$, is given by the conditions

$$R = R_+\cup(-R_+), \quad R_+\cap(-R_+) = \varnothing.$$
$$\alpha,\beta\in R_+ \text{ with } \alpha+\beta\in R \Longrightarrow \alpha+\beta\in R_+.$$

Note that the simple roots are precisely those roots which are not the sum of two (nonzero) positive roots. A Weyl chamber defines a positive root system and vice-versa.

Definition. Let R_+ be a positive root system. A root $\lambda\in R_+$ is called simple if it is not the sum of any two other roots in R_+. Let C be the Weyl chamber determined by the positive root system R_+. Then the set of simple roots of R_+, $S(R_+)$, is the set of $\alpha\in R_+$ with positive projection onto C, $\langle\alpha,\lambda\rangle > 0$ for all $\lambda\in C$.

Definition. Let α_i, α_j be simple roots. Then $m_{ij} = 0,1,2,3$; the matrix

$$\frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle}$$

is called the Cartan matrix.

Definition. The Cartan matrix can be represented uniquely as a diagram according to the following prescription. As above, let $\mathfrak{h}$ be a Cartan subalgebra with associated simple roots S. For each simple root in S, draw one open circle. Connect the circles for roots i,j by a number of lines equal to m_{ij}. Recall from the above that $m_{ij} = 0,1,2,3$. If m_{ij} equals 2 or 3, then one of the roots is shorter; draw an arrow toward the shorter root. Note that this diagram will be connected if and only if $\mathfrak{g}$ is simple. It is called the Dynkin diagram for $\mathfrak{g}$.

4.13. Theorem. *Let* $\mathrm{Aut}\,(\mathfrak{g})$ *be the group of automorphisms of* $\mathfrak{g}$ *and let* $\mathrm{Inn}\,(\mathfrak{g})$ *be the inner automorphisms, which are by definition given by the adjoint action on* $\mathfrak{g}$. *Then we have*

1. $\mathrm{Inn}\,(\mathfrak{g})$ *is a normal subgroup of finite index in* $\mathrm{Aut}\,(\mathfrak{g})$.
2. $\mathrm{Aut}\,(\mathfrak{g})/\mathrm{Inn}\,(\mathfrak{g})$ *is the symmetry group of the Dynkin diagram of* $\mathfrak{g}$.

Proof. See [Wol80, p. 85]. □

Remark. The A_l series corresponds to the algebras $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C})$, which are trace-free complex $l+1$-dimensional matrices, with simply connected group $\mathrm{SL}_{l+1}(\mathbb{C})$. The split real form is $\mathrm{SL}_{l+1}(\mathbb{R})$. The compact form is $\mathrm{SU}\,(l+1)$.

Remark. The B_l series corresponds to the algebras $\mathfrak{g} = \mathfrak{o}_{2l+1}(\mathbb{C})$, which are antisymmetric complex $2l+1$-dimensional matrices, with simply connected group the two-sheeted cover of $\mathrm{SO}\,(2l+1;\mathbb{C})$. The split real form is $SO(l, l+1)$. The compact form is $\mathrm{SO}\,(2l+1)$.

Remark. The C_l series corresponds to the algebras $\mathfrak{g} = \mathfrak{sp}\,(l;\mathbb{C})$, which are complex $2l$-dimensional matrices annihilating the antisymmetric form $A((x,x'),(y,y')) = x^T y' - x'^T y$, with group $\mathrm{Sp}\,(l;\mathbb{C})$. The split real form is $\mathrm{Sp}\,(l;\mathbb{R})$. The compact form is $\mathrm{Sp}\,(l;\mathbb{C}) \cap U(2l)$.

Remark. The D_l series corresponds to the algebras $\mathfrak{g} = \mathfrak{o}_{2l}(\mathbb{C})$, with group $\mathrm{SO}\,(2l;\mathbb{C})$. The split real form is $SO(l,l)$. The compact form is $\mathrm{SO}\,(2l)$.

4.6 $\mathfrak{g}$ Modules

Definition. Let V be a vector space over a field K. Suppose there exists a map (multiplication), linear in both factors, satisfying

1. $\mathfrak{g} \times V \to V$, $x, v \mapsto xv$
2. $[x,y]\,v = x(yv) - y(xv)$.

Then V is called a left $\mathfrak{g}$-module. Note that every finite-dimensional $\mathfrak{g}$-module provides a representation of $\mathfrak{g}$.

Remark. Note that $\mathfrak{g}$ is itself a $\mathfrak{g}$-module, with multiplication given by the Lie bracket $[\cdot,\cdot]$. This is called the adjoint $\mathfrak{g}$-module, and the representation which it provides is called the adjoint representation of $\mathfrak{g}$.

Definition. Let $T(\mathfrak{g})$ be the tensor algebra over $\mathfrak{g}$. Let $\mathfrak{J}$ be the two-sided ideal in $T(\mathfrak{g})$ generated by the elements of the form $X \otimes Y - Y \otimes X - [X,Y]$. Define the universal enveloping algebra of $\mathfrak{g}$ to be the factor algebra

$$U\,(\mathfrak{g}) = T(\mathfrak{g})/\mathfrak{J}.$$

Definition. The root lattice is the lattice generated by the simple roots,

$$\Lambda_{\mathrm{root}} = \mathbb{Z}[S].$$

The weights are the elements of the root space given by

$$\Lambda_{\text{weight}} = \left\{ \beta \in \mathfrak{h}^* \; : \; \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \text{ for all } \alpha \in S \right\}.$$

This is a lattice as well, as indicated by the notation.

Definition. The dominant weights are defined by

$$\Lambda^+_{\text{weight}} = \left\{ \beta \in \Lambda_{\text{weight}} \; : \; \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \geq 0 \text{ for all } \alpha \in S \right\}.$$

Remark. We have already defined weights to be functionals $\beta : \mathfrak{g} \longrightarrow \mathbb{C}$ such that for some representation ρ on V there is a vector v with $(\rho(X) - \beta(X)1)^k v = 0$ for some k. The equivalence of this definition to the above is somewhat nontrivial. The set of all such vectors v was called a weight space for the β, V_β. We have

$$V_\beta = \{ v \in V \; : \; \rho(X)v = \beta(X)v \text{ for } X \in \mathfrak{h} \}.$$

Definition. Suppose (ρ, V) is an irreducible finite-dimensional representation. Then there exists a highest weight $\beta_{HW} \in \Lambda^+_{\text{weight}}$. Furthermore, V is isomorphic to a specific space, constructed as follows. Define the following

$$\rho_+ = \frac{1}{2} \sum_{\alpha \in R_+} \alpha,$$

$$\mathfrak{n}_+ = \sum_{\alpha \in R_+} \mathfrak{g}_\alpha,$$

$$\mathfrak{n}_- = \sum_{\alpha \in R_+} \mathfrak{g}_{-\alpha},$$

$$M(\beta)_\mu = \sum_{p_i \in \mathbb{Z}, p_i \geq 0, \beta - \rho_+ \sum p_i \alpha_i = \mu} [e^{p_1}_{-\alpha_1}, \ldots, e^{p_n}_{-\alpha_n}] \otimes \mathbb{C}.$$

And let

$$M(\beta) = \sum_{\nu \in \Lambda^+_{\text{root}}} M(\beta)_{\beta - \rho_+ \nu}.$$

Here $[e^{p_1}_{-\alpha_1}, \ldots, e^{p_n}_{-\alpha_n}]$ is defined to be the subalgebra of the universal enveloping algebra generated by the indicated elements. The space $M(\beta)$ is called the Verma module for β. The subalgebra $\mathfrak{n}_+$ is called the nil-radical.

Definition. Let $\overline{M}$ be the maximal proper $\mathfrak{g}$-submodule of $M(\beta)$. Define $L(\beta) = M(\beta)/\overline{M}$. Then the space V above is isomorphic to

$$V_{\beta_{HW}} \cong L(\beta_{HW} + \rho_+).$$

4.14. Theorem (Cartan). *Let $\mathfrak{g}$ be a complex reductive Lie algebra. Then the association of weights with spaces given by*

$$\beta \leftrightarrow L(\beta + \rho_+)$$

is a bijection of $\Lambda^+_{\text{weight}}$ onto the set of equivalence classes of irreducible finite-dimensional $\mathfrak{g}$-modules.

Proof. See [Wol80, p. 97]. □

Chapter 5

Lie Groups

5.1 Introduction

Definition. Define a Lie group to be a group which has the structure of a C^∞ differentiable manifold, such that the group operations are smooth. Clearly Lie groups are locally compact since they are locally Euclidean.

5.1. Theorem (Gleason-Montgomery-Zippen). *Let G be a locally Euclidean topological group which is connected. Then G admits a differentiable manifold structure making it into a Lie group.*

Proof. This is difficult. The proof constitutes an affirmative solution to Hilbert's fifth problem. [MZ55]. ☐

Definition. Define a Lie subgroup of a Lie group G to be a subgroup $H \leq G$ which is also a submanifold.

Definition. A linear group is a Lie subgroup of $\mathrm{GL}_n(K)$, where K is either $\mathbb{R}$ or $\mathbb{C}$.

Definition. A linear connected reductive group is a closed connected linear group which is stable under conjugate transpose. Inverse conjugate transpose for linear groups is called the Cartan involution.

Definition. A linear connected semisimple group is a linear connected reductive group with finite center.

Definition. A Lie group G is called simple if the following hold.

- $\dim G > 1$.
- G has finitely many connected components.
- Any proper normal subgroup of the identity component of G is finite.

Definition. A Lie group G is called reductive if the following hold.

- G has finitely many connected components.

- Some finite cover of the identity component of G is a product of simple and abelian groups.

Definition. A Lie group G is called semisimple if it is reductive and the decomposition above contains no abelian factors.

Remark. This definition of reductive is taken from Vogan [Vog87]. It is subject to debate. The following is another definition.

Definition. A Lie group G is called reductive if it has a finite-dimensional representation with discrete kernel such that the complement of any invariant subspace is invariant. Such representations are called semisimple representations.

5.2 Infinitesimal Theory

Definition. A vector field X on a Lie group G is said to be left-invariant if $d\tau_\sigma \circ X = X \circ \tau_\sigma$, where τ_σ is left-translation. We have some standard results summarized in the following theorem.

5.2. Theorem. *Let $\mathfrak{g}$ be the set of left-invariant vector fields on a Lie group G. Then*

1. $\mathfrak{g} \cong TG_e$.
2. $\mathfrak{g}$ *is a Lie algebra with* $[X, Y](f) = X(Yf) - Y(Xf)$.

Proof. See [War83, p. 85]. □

Definition. As usual, let δ denote the dual to d; $\delta f : TN^*_{f(m)} \longrightarrow TM^*_m$. A form ω on G is called left-invariant if $\delta\tau_\sigma\omega = \omega$. Left-invariant 1-forms are called Maurer-Cartan forms.

5.3. Theorem (Maurer-Cartan). *Let $\{X_1, \ldots, X_n\}$ be a basis for $\mathfrak{g}$ and let $\{\omega_1, \ldots, \omega_n\}$ be the dual basis. Then there exist constants c_{ijk} such that $[X_i, X_j] = c_{ijk}X_k$, and furthermore $d\omega_i = \sum_{j<k} c_{jki}\omega_k \wedge \omega_j$.*

Proof. See [War83, p. 89]. □

Definition. Let $\sigma \in G$. Conjugation by σ acts on the space of left-invariant vector fields. Therefore it induces an automorphism of $\mathfrak{g}$. Denote this automorphism by $\mathrm{Ad}(\sigma)$.

5.4. Theorem (Lie Modular Function). *Let G be a Lie group. Let $d\mu_R$ and $d\mu_L$ be right- and left-invariant Haar measures respectively. Then $d\mu_R(g) = c[\det(\mathrm{Ad}(g))]d\mu_L(g)$, where c is some nonzero constant.*

Proof. Let $d\theta = (\det(\mathrm{Ad}(g)))d\mu_L(g)$. Let τ and ρ be the left and right translations, and let $I(g) = \tau_g \circ \rho_{g^{-1}}$ be the conjugation map. Then

$$\begin{aligned}(\rho_{a^{-1}})^* d\theta &= \det\left(\mathrm{Ad}\left(ga^{-1}\right)\right)(\rho_{a^{-1}})^* d\mu_L(g)\\ &= \det\left(\mathrm{Ad}\left(ga^{-1}\right)\right) I(a)^* d\mu_L(g).\end{aligned}$$

And $(I(x)^* d\mu_L(g))_e = \det(\mathrm{Ad}(x))(d\mu_L(g))_e$. So

$$\begin{aligned}(\rho_{a^{-1}})^* d\theta)_e &= \det\left(\mathrm{Ad}\left(a^{-1}\right)\right)\det(\mathrm{Ad}(a))\, d\mu_L(g)_e\\ &= d\mu_L(g)_e\\ &= (d\theta)_e.\end{aligned}$$

Therefore $d\theta$ is left-invariant, and the result follows from the uniqueness of Haar measure. □

5.5. Theorem (Unimodularity Conditions).

1. *Semisimple Lie groups are unimodular.*
2. *Connected nilpotent Lie groups are unimodular.*
3. *If* $\mathrm{Ad}\,(G)$ *is compact then* G *is unimodular.*

Proof.

1. $\mathrm{Ad}\,(g)$ leaves invariant the non-degenerate Killing form. Therefore $|\det\left(\mathrm{Ad}\,(g)\right)|^2 = 1$. Apply the previous theorem.
2. G is nilpotent; if $X \in \mathfrak{g}$ then $\mathrm{ad}\,(X)$ is nilpotent. Therefore $\mathrm{Tr}\,(\mathrm{ad}\,(X)) = 0$. But $\det\left(e^M\right) = e^{\mathrm{Tr}(M)}$, so $\det\left(\mathrm{Ad}\,(\exp X)\right) = 1$. Apply the previous theorem.
3. $\{|\det\left(\mathrm{Ad}\,(G)\right)| \,:\, g \in G\}$ is a subgroup of $\mathbb{R}^*$. If it is compact, then it is equal to $\{1\}$. Apply the previous theorem.

□

Definition. Let G be a connected Lie group; let σ be an involutary automorphism of G, so $\sigma^2 = 1$. Let H be a closed subgroup of G and consider the homogeneous space G/H. G/H is called a symmetric space if $\sigma(g) = g \Longleftrightarrow g \in H$.

Remark. In the above situation, the Lie algebra of G will split $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\sigma(k) = k, k \in \mathfrak{k}$, $\sigma(p) = -p, p \in \mathfrak{p}$, and we have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Definition. Define the rank of a symmetric space to be the dimension of the maximal abelian subgroup of $\mathfrak{p}$ in the above split.

Remark. $G = \mathrm{SO}\,(n+1)$, $H = \mathrm{SO}\,(n)$,

$$\sigma(g) = \begin{pmatrix} -1 & 0 \\ 0 & 1_{n\times n} \end{pmatrix} g \begin{pmatrix} -1 & 0 \\ 0 & 1_{n\times n} \end{pmatrix}^{-1}.$$
$$G/H \cong S^n$$
$$\mathrm{rank}\,(G/H) = 1.$$

Remark. The derivative $p_* : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is an isomorphism of Lie algebras, so each connected Lie group with a given algebra $\mathfrak{g}$ is obtained by factoring its universal covering group by some discrete subgroup of the center of the covering group. So we see that, up to taking covering groups, the structure of a Lie group is determined by its Lie algebra. Furthermore, Lie algebras will play a central role in the representation theory of semisimple Lie groups.

Remark. Recall that (possibly by passing to an extension field) we could find decompositions of $\mathfrak{g}$, relative to any Cartan subalgebra $\mathfrak{h}$.

$$[\mathfrak{h}, \mathfrak{h}] = 0$$
$$\mathfrak{g} = \mathfrak{g}_{\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{\alpha_k}$$
$$\mathrm{ad}\,(\mathfrak{h}) \cdot X_\alpha = \alpha(\mathfrak{h}) X_\alpha, \quad X_\alpha \in \mathfrak{g}_\alpha.$$

The will depend on the choice of $\mathfrak{h}$. However, we have the following important result, which is the starting point for representation theory in the noncompact case.

5.6. Theorem (Chevalley). *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$.*

1. *If $\mathfrak{g}$ is complex, then any two Cartan subalgebras are $\mathrm{Ad}\,(G)$-conjugate.*
2. *If $\mathfrak{g}$ is real, then there are finitely many $\mathrm{Ad}\,(G)$-conjugacy classes of Cartan subalgebras.*

Proof. I do not know how to prove this. The statement is from Wolf's lectures [Wol80, p. 79]. □

Remark. The following theorem provides a direct prove of the above for the compact case. The proof shows that this case is similar to the complex case.

5.7. Theorem. *Let G be a compact and connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be Cartan subalgebras. Then there exists a $g \in G$ such that $\mathrm{Ad}\,(g)\,\mathfrak{h}_1 = \mathfrak{h}_2$.*

Proof. Following Bott, we first prove a statement about the orbits of $\mathrm{Ad}\,(G)$. Let $\mathcal{O}_Y = \mathrm{Ad}\,(G)\,Y$, $Y \in \mathfrak{g}$. Then $\mathcal{O}_Y$ intersects $\mathfrak{h}$ in a finite non-empty set of points. To see this, define $f : \mathcal{O}_Y \longrightarrow \mathbb{R}$ by $f(Z) = (Z, X)$, where $X \in \mathfrak{g}$ is such that $\mathfrak{h} = \mathrm{Lie}(\{g \in G \,:\, \mathrm{Ad}\,(g)\,X = 0\})$. $\mathcal{O}_Y$ is compact so f achieves a minimum, say at $Y \in \mathcal{O}_Y$. Now

$$\frac{d}{dt} f(\mathrm{Ad}\,(\exp tZ)\,(Y))|_{t=0} = 0,$$

since Y is the minimum. But the left-hand side equals $([Z, Y], X)$. Therefore $([Z, Y], X) = 0$ for all $Z \in \mathfrak{g}$, and so $(Y, [X, Z]) = 0$, so $Y \in \mathfrak{h}$.

Clearly $\mathcal{O}_Y$ meets $\mathfrak{h}$ perpendicularly, so they meet in a discrete set of points. But since they meet perpendicularly, the set is finite. So we see that $\mathcal{O}_Y$ intersects $\mathfrak{h}$ in a finite non-empty set.

Now let $\mathfrak{h}_1 = \mathrm{Lie}(\{g \in G \,:\, \mathrm{Ad}\,(g)\,X_1 = 0\})$, and similarly for $\mathfrak{h}_2$. Then there exists $g \in g$ such that $\mathrm{Ad}\,(g)\,X_2 \in \mathfrak{h}_1$, by the intersection result. so $\mathrm{Ad}\,(g)\,\mathfrak{h}_2 = \mathfrak{h}_1$. □

Remark. So we see that a compact connected Lie group has essentially one Cartan subalgebra, up to $\mathrm{Ad}\,(G)$-conjugacy, which is a trivial difference. However, the same is not true for non-compact groups. For example, take $G = \mathrm{SL}_2\,(\mathbb{R})$. Then one Cartan subgroup is

$$H_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,:\, a \in \mathbb{R}, a \neq 0 \right\}.$$

The other one (there are only two) is

$$H_2 = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \,:\, \theta \in T \right\}.$$

5.3 Decomposition Theory

Remark. There are several useful ways to decompose groups into simpler pieces. These methods revolve around the extraction of maximal compact subgroups.

Remark. Let G have finitely many components. Then every compact subgroup of G is contained in a maximal compact subgroup. This follows simply from the local compactness of G. If K is a maximal compact subgroup then G/K is a symmetric space; the involution which fixes K is called the Cartan involution. G/K is a complete simply connected Riemannian manifold of negative curvature.

Remark. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $R^+_{\mathfrak{h}}$ be a positive root system for $\mathfrak{h}$. We can define a nilpotent subalgebra of $\mathfrak{g}$ by

$$\mathfrak{n} = \sum_{\nu \in R^+_{\mathfrak{h}}} \mathfrak{g}_\nu .$$

Integrate $\mathfrak{n}$ to obtain a connected, simply connected nilpotent subgroup of G, $N = \exp_G \mathfrak{n}$. Let $A = \exp_G \mathfrak{h}$.

5.8. Theorem (Iwasawa). *Let $\psi : K \times A \times N \longrightarrow G$ be the map $\psi(k, a, n) = kan$. Then ψ is a diffeomorphism, $G \cong K \times A \times N$.*

Proof. $\square$

Definition. Let M be the centralizer of A in K, $M \subseteq K$. Then a minimal parabolic subgroup for the given decomposition is the subgroup

$$P = MAN \subset G.$$

This decomposition is called the Levy-Langlands decomposition. Minimal parabolic subgroups are used in the construction of the principal series of representations.

5.4 Topology of Compact Lie Groups

Remark. To begin, we will consider the classical groups $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$.

Definition. The infinite classical groups are defined by

$$\mathrm{O}(\infty) = \bigcup_{n\geq 1} \mathrm{O}(n), \quad \mathrm{SO}(\infty) = \bigcup_{n\geq 1} \mathrm{SO}(n),$$
$$\mathrm{U}(\infty) = \bigcup_{n\geq 1} \mathrm{U}(n), \quad \mathrm{SU}(\infty) = \bigcup_{n\geq 1} \mathrm{SU}(n),$$
$$\mathrm{Sp}(\infty) = \bigcup_{n\geq 1} \mathrm{Sp}(n),$$

where the topology is the inductive limit topology. Recall that $X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset X$ gives X as an inductive limit if the inclusion maps $X_m \to X_{m+k}$ are continuous and any convex set $V \subset X$ is a neighbourhood of $0 \in X$ if and only if $V \cap X_n$ is a neighbourhood of $0 \in X_n$ for all n.

Definition. The Steifel spaces are the following coset spaces, which can be identified to spaces of k-tuples.

$$\begin{aligned}
&V_k(\mathbb{R}^n) = \mathrm{O}(n)/\mathrm{O}(n-k)\,[= \mathrm{SO}(n)/\mathrm{SO}(n-k), \quad k<n] \\
&V_k(\mathbb{C}^n) = \mathrm{U}(n)/\mathrm{U}(n-k)\,[= \mathrm{SU}(n)/\mathrm{SU}(n-k), \quad k<n] \\
&V_k(\mathbb{H}^n) = \mathrm{Sp}(n)/\mathrm{Sp}(n-k) \\
&V_n(\mathbb{R}^n) = \mathrm{O}(n), \quad V_n(\mathbb{C}^n) = \mathrm{U}(n), \quad V_n(\mathbb{H}^n) = \mathrm{Sp}(n).
\end{aligned}$$

Definition. The Grassmanian manifolds are obtained from the Steifel manifolds by identification of k-tuples with planes.

$$\begin{aligned}
G_k(\mathbb{R}^n) &= \mathrm{O}(n)/(\mathrm{O}(n-k)\times\mathrm{O}(k)) \\
G_k(\mathbb{C}^n) &= \mathrm{U}(n)/(\mathrm{U}(n-k)\times\mathrm{U}(k)) \\
G_k(\mathbb{H}^n) &= \mathrm{Sp}(n)/(\mathrm{Sp}(n-k)\times\mathrm{Sp}(k)).
\end{aligned}$$

Similar define oriented Grassmanians for $\mathbb{R}$ and $\mathbb{C}$ by

$$\begin{aligned}
SG_k(\mathbb{R}^n) &= \mathrm{O}(n)/(\mathrm{O}(n-k)\times\mathrm{SO}(k)) \\
SG_k(\mathbb{C}^n) &= \mathrm{U}(n)/(\mathrm{U}(n-k)\times\mathrm{SU}(k))
\end{aligned}$$

Remark. Some examples.

- $G_1(\mathbb{R}^n) = \mathbb{R}P^{n-1}$.
- $G_1(\mathbb{C}^n) = \mathbb{C}P^{n-1}$.
- $G_1(\mathbb{H}^n) = \mathbb{H}P^{n-1}$.
- $SG_1(\mathbb{R}^n) = S^{n-1}$.
- $SG_1(\mathbb{H}^n) = S^{2n-1}$.

Remark. The coset spaces are naturally bundles, and we have the following short sequences.

$$\begin{aligned}
\mathrm{O}(n) \to \mathrm{O}(n+1) = V_{n+1}\left(\mathbb{R}^{n+1}\right) \to \mathrm{O}(n+1)/\mathrm{O}(n) = V_1\left(\mathbb{R}^{n+1}\right) = S^n \\
\mathrm{U}(n) \to \mathrm{U}(n+1) = V_{n+1}\left(\mathbb{C}^{n+1}\right) \to \mathrm{U}(n+1)/\mathrm{U}(n) = V_1\left(\mathbb{C}^{n+1}\right) = S^{2n+1}.
\end{aligned}$$

5.9. Theorem (Stable Homotopy).

1. *Let* $i \le n-2$. *Then* $\pi_i(\mathrm{O}(n)) \cong \pi_i(\mathrm{O}(n+q))$, $\pi_i(\mathrm{SO}(n)) \cong \pi_i(\mathrm{SO}(n+q))$.
2. *Let* $i \le 2n-1$. *Then* $\pi_i(\mathrm{U}(n)) \cong \pi_i(\mathrm{U}(n+q))$, $\pi_i(\mathrm{SU}(n)) \cong \pi_i(\mathrm{SU}(n+q))$.

Proof. [Hus, p.82] If we have a fibering with projection $p: E \longrightarrow B$ and fiber F, then there is an exact homotopy sequence

$$\cdots \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F)\,\pi_{n-1}(E) \to \cdots .$$

Therefore we have the exact sequences

$$\pi_{i+1}\left(S^{n}\right) \to \pi_{i}\left(\mathrm{O}\left(n\right)\right) \to \pi_{i}\left(\mathrm{O}\left(n+1\right)\right) \to \pi_{i}\left(S^{n}\right)$$
$$\pi_{i+1}\left(S^{2n+1}\right) \to \pi_{i}\left(\mathrm{U}\left(n\right)\right) \to \pi_{i}\left(\mathrm{U}\left(n+1\right)\right) \to \pi_{i}\left(S^{2n+1}\right),$$

and similarly for $\mathrm{SO}\left(n\right)$ and $\mathrm{SU}\left(n\right)$. Clearly if i is small enough, then the sphere homotopies will vanish. This establishes the result for $q=1$. For $q>1$ factor the inclusions

$$\mathrm{O}\left(n\right) \to \mathrm{O}\left(n+1\right) \to \cdots \to \mathrm{O}\left(n+q\right)$$
$$\mathrm{U}\left(n\right) \to \mathrm{U}\left(n+1\right) \to \cdots \to \mathrm{U}\left(n+q\right).$$

□

5.10. Theorem (Steifel Homotopy).

1. *If* $i \leq m-1$*, then* $\pi_{i}\left(V_{k}\left(\mathbb{R}^{k+m}\right)\right)$*=0.*
2. *If* $i \leq 2m$*, then* $\pi_{i}\left(V_{k}\left(\mathbb{C}^{k+m}\right)\right)$*=0.*

Proof. Again consider the exact homotopy sequence; for example

$$\cdots \to \pi_{i}\left(\mathrm{O}\left(m\right)\right) \to_{\alpha} \pi_{i}\left(\mathrm{O}\left(m+k\right)\right) \to \pi_{i}\left(V_{k}\left(\mathbb{R}^{k+m}\right)\right) \to \pi_{i-1}\left(\mathrm{O}\left(m\right)\right) \to_{\beta} \cdots .$$

By the previous result β is an isomorphism. Clearly then $\pi_{i}\left(V_{k}\left(\mathbb{R}^{k+m}\right)\right)=0$ since α is onto. □

5.11. Theorem (Group Classifying Spaces). *The fibration* $V_{k}\left(\mathbb{R}^{\infty}\right) \to G_{k}\left(\mathbb{R}^{\infty}\right)$ *is a universal bundle for* $\mathrm{O}\left(k\right)$*. Similarly* $V_{k}\left(\mathbb{C}^{\infty}\right) \to G_{k}\left(\mathbb{C}^{\infty}\right)$ *is a universal bundle for* $\mathrm{U}\left(k\right)$*;* $V_{k}\left(\mathbb{R}^{\infty}\right) \to SG_{k}\left(\mathbb{R}^{\infty}\right)$ *is universal for* $\mathrm{SO}\left(k\right)$*;* $V_{k}\left(\mathbb{C}^{\infty}\right) \to SG_{k}\left(\mathbb{C}^{\infty}\right)$ *is universal for* $\mathrm{U}\left(k\right)$*.*

Proof. This follows from the above stability results together with the definition of universal bundle. See [Hus, p.83]. □

Remark. At this point is relatively straightforward to calculate some homotopy groups for the classical groups. We use the fact that the fibration $\mathrm{SO}\left(n\right) \to \mathrm{O}\left(n\right) \to \mathbb{Z}_{2}$ gives the exact sequence $0=\pi_{i+1}\left(\mathbb{Z}_{2}\right) \to \pi_{i}\left(\mathrm{SO}\left(n\right)\right) \to \pi_{i}\left(\mathrm{O}\left(n\right)\right) \to \pi_{i}\left(\mathbb{Z}_{2}\right)$. Similarly $0=\pi_{i+1}\left(S^{1}\right) \to \pi_{i}\left(\mathrm{SU}\left(n\right)\right) \to \pi_{i}\left(\mathrm{U}\left(n\right)\right) \to \pi_{i}\left(S^{1}\right)$.

5.12. Theorem.

1. *If* $i \geq 1$ *then* $\pi_{i}\left(\mathrm{SO}\left(n\right)\right) \cong \pi_{i}\left(\mathrm{O}\left(n\right)\right)$*.*
2. *If* $i \geq 2$ *then* $\pi_{i}\left(\mathrm{SU}\left(n\right)\right) \cong \pi_{i}\left(\mathrm{U}\left(n\right)\right)$*.*
3. *If* $i=1$ *then* $\pi_{1}\left(\mathrm{SU}\left(n\right)\right) \to \pi_{1}\left(\mathrm{O}\left(n\right)\right) \to \mathbb{Z} \to 0$*.*

Proof. An easy consequence of the sequences in the above remark. □

5.13. Theorem.

1. $\pi_{1}\left(\mathrm{O}\left(n\right)\right)=\pi_{1}\left(\mathrm{SO}\left(n\right)\right)=\mathbb{Z}_{2}$*, for* $n \geq 3$*.*
2. $\pi_{1}\left(\mathrm{O}\left(2\right)\right)=\pi_{1}\left(\mathrm{SO}\left(2\right)\right)=\mathbb{Z}$*.*

3. $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$, $n \geq 1$.

4. $\pi_1(\mathrm{SU}(n)) = 0$, $n \geq 1$.

5. $\pi_1(\mathrm{Sp}(n)) = 0$, $n \geq 1$.

Proof. For $\mathrm{U}(1)$, $\mathrm{SU}(2)$, and $\mathrm{Sp}(1) = S^3$, simply calculate and then use the stable homotopy result. For $\mathrm{O}(2)$, calculate. For $\mathrm{O}(3)$, and thus $\mathrm{O}(n \geq 3)$ by stability, use the homeomorphism $\mathrm{SO}(3) \cong \mathbb{R}P^3$. So we see that $\pi_1(G)$ is a trivial consequence of stable homotopy. ☐

5.14. Theorem. *If G is one of $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, then $\pi_2(G) = 0$.*

Proof. Use $\mathrm{SO}(3) \to \mathrm{SO}(4) \to S^3$ to get the exact sequence $0 = \pi_2(\mathrm{SO}(3)) \to \pi_2(\mathrm{SO}(4)) \to \pi_2(S^3) = 0$. Therefore $\pi_2(\mathrm{SO}(4)) = \pi_2(\mathrm{O}(4)) = 0$. By stability then $\pi_2(\mathrm{SO}(n)) = \pi_2(\mathrm{O}(n)) = 0$ for $n \geq 4$. We already noted that $\pi_2(\mathrm{SO}(3)) = 0$. That proves the result for the $\mathrm{SO}(\cdot)$ series. The rest are even easier. ☐

Remark. This result on $\pi_2(G)$ is actually true for any compact Lie group G, though the general proof is significantly more complicated than the above.

5.15. Theorem.

1. $\pi_3(\mathrm{U}(n)) = \pi_3(\mathrm{SU}(n)) = \mathbb{Z}$, $n \geq 2$.

2. $\pi_3(\mathrm{Sp}(n)) = \mathbb{Z}$, $n \geq 1$.

Proof. As noted before $\mathrm{SU}(2) \cong \mathrm{Sp}(1) \cong S^3$. So $\pi_3(\mathrm{U}(2)) = \pi_3(\mathrm{SU}(2)) = \mathbb{Z} = \pi_3(\mathrm{Sp}(1))$. Then the result follows from stability. ☐

5.16. Theorem (Hopf). *Let G be a compact connected Lie group. Then for some integers $\{k_\alpha\}$ depending on G we have*

$$\mathrm{H}^*(G, \mathbb{R}) \cong \mathrm{H}^*\left(\prod_\alpha S^{2k_\alpha - 1}, \mathbb{R}\right).$$

Proof. I do not know where to find the proof. The statement is from some lectures by Bott. [Bot77]. ☐

Remark. The set $\{k_\alpha\}$ is called the set of exponents of G. We have

$$\begin{aligned}
\mathrm{H}^*(\mathrm{SU}(n), \mathbb{R}) &\cong \mathrm{H}^*(\mathrm{SU}(n-1), \mathbb{R}) \otimes \mathrm{H}^*\left(S^{2n-1}, \mathbb{R}\right) \\
&\cong \mathrm{H}^*\left(S^3, \mathbb{R}\right) \otimes \mathrm{H}^*\left(S^5, \mathbb{R}\right) \otimes \cdots \otimes \mathrm{H}^*\left(S^{2n-1}, \mathbb{R}\right), \\
\mathrm{H}^*(\mathrm{Sp}(n), \mathbb{R}) &\cong \mathrm{H}^*\left(S^3, \mathbb{R}\right) \otimes \mathrm{H}^*\left(S^7, \mathbb{R}\right) \otimes \cdots \otimes \mathrm{H}^*\left(S^{4n-1}, \mathbb{R}\right).
\end{aligned}$$

These are apparently easy to prove using the obvious fibrations and induction.However, the induction fails for $\mathrm{SO}(n)$, and the results there are more complicated. See Ref. [Bot77]. Bott appears to prove at least parts of the general statement, using some nifty Morse theoretic construction.

Remark. The universal covering group of a Lie group exists, as it does for a general topological group. Recall that if $\tilde{G}$ is the universal covering group of G, then

$$G \cong \tilde{G}/N,$$

for some discrete normal subgroup N; N is the kernel of a group homomorphism $p : \tilde{G} \longrightarrow G$, $N = \ker(p)$, and $N \cong \pi_1(G)$, which we can at the very least calculate using an exact sequence [there are probably easier ways].

Remark. Consider the map $\tilde{g} \mapsto \tilde{g}n\tilde{g}^{-1}$ for $n \in N$. This is a map $\tilde{G} \to N$ for each n; it is continuous, but N is discrete so it is a constant map. Therefore $\tilde{g}n\tilde{g}^{-1}$ is independent of $\tilde{g}$, so N is a subgroup of the center of $\tilde{G}$, $\mathcal{Z}\left(\tilde{G}\right)$. Therefore, in particular, $N = \pi_1(G)$ is *abelian*.

Chapter 6

Representations of Lie Groups

6.1 Compact Lie Groups: Weyl Formulae

Remark. We have seen the following facts for compact topological groups.

- From the general theory of locally compact groups, we know that irreducible representations are completely determined by their characters.
- From the theory of compact topological groups, we know that the representations to be considered are irreducible, automatically unitary, and automatically finite-dimensional. Their characters are just $\mathrm{Tr}\,(\rho(\cdot))$. The Peter-Weyl theorem further asserts that all of these representations occur in the decomposition of the regular representation, $L^2\,(G; dg) \cong \oplus V_i \otimes V_i^*$, which is the existence theorem for harmonic analysis on G.

The actual construction of representations is left open in general. In the case of compact Lie groups, Weyl's theory solves this problem explicitly. Compact and acceptable Lie groups obey the Weyl character formula, which determines the characters of all the irreducible representations, The dimensions of the representations are determined by the Weyl dimension formula, and the explicit structure of the representations is given in terms of Verma modules.

Remark. Recall the definition

$$\rho_+ = \frac{1}{2} \sum_{\alpha \in R_+} \alpha,$$

where R_+ is a positive root system for $\mathfrak{g}$. See 4.6.

Definition. Let T be a Cartan subgroup of G, with associated Cartan subalgebra $\mathfrak{t}$. G is called acceptable if there is a character of T, ξ_{ρ_+}, satisfying

$$\xi_{\rho_+}(\exp Y) = e^{\rho_+(Y)}, \quad Y \in \mathfrak{t}.$$

Remark. Characters χ can be identified with linear functions $\lambda(\chi)$, $\chi(\exp Y) = e^{\lambda(\chi)(Y)}$. Weyl discovered that if G is compact, simply connected, and semisimple,then its universal covering group is compact and acceptable. This makes the assumption of acceptability in the following theorem interesting.

6.1. Theorem (Weyl Character Formula). *Let G be a compact, acceptable, and connected Lie group. Let R_+ be a positive root system. Then the irreducible characters of G are in bijective correspondence with the positive characters of T.*

Let χ be a positive character of T, then the corresponding irreducible character of G, Θ_χ, is given by

$$\Theta_\chi = \left[\sum_{w \in \mathrm{Weyl}(G)} \det(w)\,(\chi \circ w) \right] \frac{1}{\Delta^+},$$

where

$$\Delta^+ = \xi_{\rho_+}^{-1} \prod_{\alpha \in R^+} (\xi_\alpha - 1).$$

Furthermore, ξ_{ρ_+} is a positive character, $\Theta_{\xi_{\rho_+}}$ is the trivial character, and

$$\Delta^+ = \sum_{w \in \mathrm{Weyl}(G)} \det(w)\,(\xi_{\rho_+} \circ w).$$

Proof. The character formula can be proven in the general algebraic context. It is a moderately long calculation. See [Jac62, p. 249]. □

6.2. Theorem (Weyl Dimension Formula). *Let G be as above. Let ρ, V_λ be the representation on the $\mathfrak{g}$-module $V_\lambda \cong L(\lambda + \rho_+)$, corresponding to the highest weight λ. Then*

$$\dim V_\lambda = \prod_{\alpha \in R_+} \frac{\langle \lambda + \rho_+, \alpha \rangle}{\langle \rho_+, \alpha \rangle}.$$

Proof. See [Wol80][Jac62, p. 256]. □

Remark. The dimension formula can be expressed in a very explicit way. Let α be any positive root. Define coefficients k_i by expressing α as a linear combination of the fundamental roots,

$$\alpha = \sum_{i=1}^{l} k_i \alpha_i.$$

These coefficients k_i are integral and non-negative. Also, express the dominant integral weight λ as a linear combination of the fundamental weights,

$$\lambda = \sum_{i=1}^{l} m_i w_i.$$

The m_i are also integral and non-negative. Finally, given any positive root α define $c(\alpha)$ by $\langle \alpha, \alpha \rangle = c(\alpha)\,\langle \alpha_0, \alpha_0 \rangle$, where α_0 is a root of minimal length. From the constraint on the Cartan matrix we know that $c(\alpha)$ can equal 1,2, or 3. Furthermore, $c(\alpha)$ can equal 3 only for the case G_2. Then we have

$$\dim V_\lambda = \prod_{\alpha \in R_+} \frac{\sum_{i=1}^{l} k_i(\alpha) c(\alpha)(m_i + 1)}{\sum_{i=1}^{l} k_i(\alpha) c(\alpha)}.$$

6.2 Compact Lie Groups: Borel-Weil Theory

Remark. Borel-Weil theory provides explicit realizations of the representations for compact Lie groups as spaces of holomorphic sections of explicit vector bundles.

Definition. Let G be a compact Lie group with maximal torus T. Let $\mathfrak{n}_+$ be the nil-radical of $\mathfrak{g}$, with conjugate subalgebra $\mathfrak{n}_-$. Define the Borel subalgebras

$$\begin{aligned} \mathfrak{b} &= \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}_+, \\ \mathfrak{b}_- &= \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}_-. \end{aligned}$$

Definition. Let G be a compact Lie group with complexification $G_{\mathbb{C}}$. Define the Borel subgroup of $G_{\mathbb{C}}$, $B = HN_- \subset G_{\mathbb{C}}$, where $H = \exp(\mathfrak{t}_{\mathbb{C}})$, $N_- = \exp(\mathfrak{n}_-)$.

Definition. Let $\lambda \in \Lambda^+_{\text{weight}}$. Define the homomorphism $e^\lambda : B \longrightarrow \mathbb{C}_\times$ by

$$e^\lambda(\exp(t+n)) = e^{\lambda(t)}, \quad t \in \mathfrak{t}_{\mathbb{C}}, n \in \mathfrak{n}_-.$$

Then e^λ is a holomorphic map and it is a representation of B on $\mathbb{C}$. Let $\mathcal{L}_\lambda$ be the associated holomorphic line bundle $\mathcal{L}_\lambda$, with base $G_{\mathbb{C}}/B$. We are especially interested in sections of this bundle, which we can identify with complex functions on $G_{\mathbb{C}}$ which transform by the representation e^λ under translations by B.

6.3. Theorem (Borel-Weil). *Let $\mathcal{H}^0(\mathcal{L}_\lambda)$ denote the space of holomorphic sections of $\mathcal{L}_\lambda$. Then $G_{\mathbb{C}}$ acts on $\mathcal{H}^0(\mathcal{L}_\lambda)$ by the irreducible representation of highest weight λ.*

Proof. There is a detailed discussion of this construction in [Vog87, ch. 2]. For a short proof see [Wol80, p. 106]. $\square$

6.3 What is a Character?

Remark. So the representation theory for compact Lie G is solved. All the questions are reduced to certain technical calculations using the results of the Weyl theory. The noncompact Lie case is much more involved. In particular, the characters must be constructed, and we know that in general they will be distributional. Infinite-dimensional representations will be indispensable. Methods for construction of representations will be important. In particular, induction from subgroups will enter.

Remark. We have seen the importance of characters. Recall that our discussions of topological groups showed that an irreducible representation of a locally compact group is determined up to equivalence by its character. Therefore it will be important to construct the characters as explicitly as possible. This construction is due to Harish-Chandra, and constitutes one of the great contributions to mathematics in the twentieth century. The basic goals for noncompact Lie groups are then

- Determine $\widehat{G}$.
- Determine the Plancherel measure $\widehat{\mu}$.
- Find an analogue of the Weyl character formula.

Remark. What do the characters look like for infinite-dimensional representations? To get some idea, consider $G = S^1$. Let ρ be the regular representation, acting on $\mathcal{H} = L^2(G; dg)$. Let $\{e_n\}$ be the basis $\left\{e^{in\theta} : n \in \mathbb{Z}\right\}$ of $\mathcal{H}$. Then G acts on $\mathcal{H}$ by $\rho(e^{i\theta})e_n = e^{-n\theta}e_n$. We would like to define the character to be

$$\chi_\rho(e^{i\theta}) = \operatorname{Tr}\left(\rho(e^{i\theta})\right) = \sum_n e^{in\theta}.$$

This converges weakly, $\chi_\rho(e^{i\theta}) = \delta(\theta)$. So apparently the characters of infinite-dimensional representations may be distributions. But we would also like to check that nothing too weird happens for representations other than the regular representations.

Let ρ be an arbitrary unitary representation of $G = S^1$, acting on $\mathcal{H}$. Let $\mathcal{H}_n \subset \mathcal{H}$ be the subspace on which ρ acts by multiplication by $e^{in\theta}$. Then we write

$$\mathcal{H} = \oplus_n \mathcal{H}_n.$$

Each $\mathcal{H}_n$ decomposes into copies of the basic representation space on which ρ acts by multiplication by $e^{in\theta}$, $\mathcal{H}_n = m_n V_n$, so $\mathcal{H} = \oplus_n m_n V_n$. Each V_n is finite-dimensional. Suppose that $m_n = \mathcal{O}(n^k)$ for some k. Then we can define

$$\chi_\rho(e^{i\theta}) = \sum_n m_n e^{in\theta}.$$

By the assumption on m_n, this converges weakly to a distribution.

So we see that not all representations of $G = S^1$ have a nice character. But the ones for which the multiplicities $\{m_n\}$ do not grow quickly will have characters which are distributions on $C^\infty(G)$. We will see that the required control of the characters is obtainable when G is connected, semisimple, and has finite center [which S^1 does not].

This useful look at the circle group was provided by Atiyah.

6.4 Harish-Chandra Program; Analytic Vectors; From $\mathfrak{g}$ to G

Remark. The Harish-Chandra program is a systematic approach to linearization of the representation theory, reducing the questions about G to questions about $\mathfrak{g}$. Because the representations are infinite-dimensional in general, certain problems must be overcome. Starting with a representation of G on a space V, we will consider the representation of a maximal compact subgroup K. The basic tool in the semisimple case is to show that this picks out a set of vectors on which $\mathfrak{g}$ acts. This set of vectors will be called the set of K-finite vectors, and passing to this set will reduce the representation theory to algebra. The detailed description of this procedure is given in the following. Note that this procedure is known to be inadequate without the assumption of semisimplicity.

Remark. Consider the simple case of a finite-dimensional representation (ρ, V), $\dim V < \infty$. Clearly we can get a representation of $\mathfrak{g}$, acting on V,

$$\rho_*(X)v = \frac{d}{dt}\rho(\exp tX)v|_{t=0}, \quad v \in V.$$

To go backwards, from ρ_* to ρ, assume G is simply connected. Then there is a unique ρ such that

$$\rho(\exp X) = \exp(\rho_*(X)).$$

So the finite-dimensional case allows a simple relation between the representations of $\mathfrak{g}$ and G, where it may be necessary to take a covering space of G.

Definition. Let V be a Banach space carrying a representation ρ of G. Define the set of differentiable vectors

$$\begin{aligned} V \supseteq V^\infty &= \{v \in V \;:\; g \mapsto \rho(g)v \text{ is in } C^\infty(G)\} \\ &= \{v \in V \;:\; g \mapsto \langle w, \rho(g)v\rangle \text{ is in } C^\infty(G) \text{ for all } w \in V^*\}. \end{aligned}$$

Definition. Introduce a representation of $\mathfrak{g}$ on V^∞ by

$$\rho_*(X)v = \frac{d}{dt}\rho(\exp tX)v|_{t=0}, \quad v \in V^\infty.$$

Let the Garding space of V be the space

$$V_0^\infty = \operatorname{Span}\{\rho(f)v \;:\; f \in C_0^\infty(G), v \in V\}$$

where $\rho(f)v$ is well-defined for $f \in C_0^\infty(G)$ by

$$\rho(f)v \equiv \int_G f(g)\rho(g)v\, dg.$$

6.4. Theorem. *$V_0^\infty \subseteq V$ and V_0^∞ is dense in V.*

Proof. Let $w \in V_0^\infty$. So we have $w \in \operatorname{Span}\{\rho(f)v\}$. Without loss let $w = \rho(f)v$ for some $f \in C_0^\infty(G)$, $v \in V$. Then

$$\begin{aligned} \rho(g)w = \rho(g)\rho(f)v &= \int_G f(g')\rho(g)\rho(g')v dg' \\ &= \int_G f(g^{-1}g')\rho(g')v dg'. \end{aligned}$$

Since $f \in C_0^\infty(G)$, we can differentiate with respect to g under the integral, and so $w \in V_0^\infty$.

Now we can prove the density result. Let $v \in V$. Consider the sequence $\{\rho(f_n)v\}$ where $\{f_n\}$ is an approximate identity.

$$\begin{aligned} \|\rho(f_n)v - v\| &\leq \int_{\operatorname{supp}(f_n)} |f_n(g)|\, \|\rho(g)v - v\|\, dg \\ &\leq C \sup_{g \in \operatorname{supp}(f_n)} \|\rho(g)v - v\| \\ &\to 0. \end{aligned}$$

Therefore $\rho(f_n)v \to v$. $\square$

Definition. The space of analytic vectors,V^ω, is given by

$$V \supseteq V^\infty \supseteq V^\omega = \{v \in V \;:\; g \mapsto \langle w, \rho(g)v\rangle \text{ is } C^\omega \text{ on } G \text{ for all } w \in V^*\}\,.$$

6.5. Theorem (Nelson). *V^ω is dense in V.*

Proof. Let $v \in V$. If we can find an approximate identity $\{f_n\}$ consisting of analytic functions, the smearing with $\{f_n\}$ will produce the sequence of approximants that we want, similar to the smearing with $C_0^\infty(G)$ functions in the Garding construction. In the analytic case the f_n cannot be compactly supported, but they must be chosen so that they die sufficiently rapidly.

Let X_i be the generators of right translations on $L^2(G; dg)$. Let Δ be the unique self-adjoint extension of the operator $X_1^2 + X_2^2 + \cdots + X_n^2$, where we can take the domain to be $C_0^\infty(G)$. For $f \in C_0^\infty(G)$, let

$$\phi(t, g) = (\exp(t\Delta)f)(g),$$

so $\phi(t, g)$ is a solution of the heat equation on G. It is a simple exercise to show that, if $d(g)$ is the geodesic distance to the identity, $1 \in G$, then

$$\langle \exp(sd(g))\phi, \phi\rangle \leq \exp(\frac{1}{2}s^2 d(g))\,\langle \exp(sd(g))f, f\rangle\,.$$

Since Δ is an elliptic operator with analytic coefficients, the solutions of the heat equation are analytic functions for $t > 0$. So if $f \in C_0^\infty(G)$ then the following integral representation defines an analytic function for $t > 0$, $v \in V$,

$$F_{f,v}(t, g) = \int_G [\exp(t\Delta)(f)](g'^{-1}g)\rho(g')v\,dg'.$$

Let $\{f_n\}$ be an approximate identity and let $\{t_n\} = \{1, 1/2, 1/3, \ldots\}$. Take the diagonal sequence as $f_n \to \delta$, $t_m \to 0$, and so

$$\|F_{f_n,v}(t_n, \cdot) - v(\cdot)\| \to 0.$$

But each $F_{\ldots}(\ldots)$ is in $C^\omega(G)$, and $v \in V$ is arbitrary, so V^ω is dense in V. $\square$

Remark. Using the heat kernel to smooth a sequence of functions is a standard trick. The proof above is a little sketchy. For more details see [BR77, p. 358].

Remark. Note that the set V^ω forms a common dense domain for the operators of the representation $\{\rho(g) \,:\, g \in G\}$.

Remark. This type of construction can be extended to semigroups and arbitrary manifolds. The method was initiated by Nelson and Garding around 1960. Harish-Chandra's 1953 proof and analytic construction were different, and somewhat more complicated.

Remark. Let G be connected, semisimple, with finite center. Recall that such a G has an Iwasawa decomposition

$$G = K \cdot A \cdot N,$$

where K is compact, A is abelian, and N is nilpotent.

Definition. Let ρ be a representation of G on a Banach space V, and suppose that G admits an Iwasawa decomposition. For a representation κ of K, $\kappa \in \widehat{K}$, let

$$V^\omega(\kappa) = V^\infty \cap V(\kappa),$$

where $V(\kappa)$ is the representation space for κ, $V(\kappa) \subseteq V$. Let

$$V_K = \sum_{\kappa \in K} V^\omega(\kappa).$$

6.6. Theorem (Harish-Chandra). *V_κ is dense in V.*

Proof. Let $v \in V$ be arbitrary. For each $\epsilon > 0$ there is a $w \in V^\omega$ with $\|v - w\| < \epsilon/2$. Since $V^\omega \subset V^\infty$, $w \in V^\infty$, and so $\sum_\kappa \kappa(K)w$ converges absolutely to w. Therefore, partial sums of $\sum_\kappa \kappa(K)w$ can be chosen arbitrarily close to v. $\square$

Definition. $v \in V$ is called K-finite if

$$\dim \operatorname{Span}\{\rho(k)v \,:\, k \in K\} < \infty.$$

In particular, ρ is called K-finite if each $V(\kappa)$ has finite dimension.

Definition. Recall some definitions that we have seen before. (ρ, V) is called topologically irreducible (TI) if V has no proper closed $\rho(G)$-invariant subspace. (ρ, V) is called topologically completely irreducible if for $T : V \longrightarrow V$ bounded, $\{v_1, \dots, v_n\} \in V$, $\epsilon < 0$, there is an $f \in C_0^\infty(G)$ with $\|(\rho(f) - T)v_i\| < \epsilon$ for $i = 1, \dots, n$. This notion is interesting because it appears in the general Schur's lemma.

6.7. Theorem (Schur). *If (ρ, V) is TCI and $T : V \longrightarrow V$ is a bounded linear transformation commuting with $\rho(G)$, then $T = c1$.*

Remark. Gelfand calls the property $T\rho(G) = \rho(G)T \Longrightarrow T = c1$ "operator irreducibility". He calls topological irreducibility "subspace irreducibility".

6.8. Theorem. *Let (ρ, V) be unitary and TI. Then it is TCI.*

Proof. Let $\mathcal{A}$ be the set of bounded linear transformations on V satisfying

$$\{T \text{ on } V \,:\, \{v_1, \dots, v_n\} \subset V, \epsilon > 0, f \in M_0(G), \|(\rho(f) - T)v_i\| < \epsilon \text{for} i = 1, \dots, n\}.$$

So $\mathcal{A}$ is the algebra of operators on V satisfying the conditions in the definition of TI. $\mathcal{A}$ is a von Neumann algebra. Also, $\rho(G) \subset \mathcal{A}$. By the assumption, using Schur's lemma, we have $\mathcal{A}^c = \mathbb{C}$. But $\mathcal{A}$ is weakly closed, so by the von Neumann bicommutant theorem $\mathcal{A}^{cc} = \mathcal{A}$. Since $A^{cc} = \mathbb{C}^c$ which is the set of all bounded operators, the algebra $\mathcal{A}$ is in fact all the bounded operators. Therefore (ρ, V) is TCI. $\square$

Definition. Let $\mathcal{Z}(\mathfrak{g})$ denote the center of the universal enveloping algebra $U(\mathfrak{g})$. Let $M(\lambda)$ be a weight space generated by a highest weight vector of weight $\lambda - \frac{1}{2}\sum_{\alpha \in R_+} \alpha$. Define the infinitesimal character to be the map giving the action of $\mathcal{Z}(\mathfrak{g})$ on $M(\lambda)$, $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}$.

6.9. Theorem. *Let (ρ, V) be a unitary and TI representation of a connected group G. Then it has an infinitesimal character.*

Proof. See [Wol80, p. 111]. □

6.10. Theorem. *Let G be connected, semisimple, with finite center. Let (ρ, V) be TI. Then ρ has an infinitesimal character if and only if it is TCI. In that case ρ is K-finite and*

$$V_K = \sum_{\kappa\in\widehat{K}} V(\kappa).$$

Proof. TCI $\Longrightarrow$infinitesimal character follows from the above. For the converse see [Wol80, p. 115]. See also [Var89, p. 143-144]. □

6.11. Corollary. *Let G be as above. Let (ρ, V) be a unitary representation of G. If $f \in L^1(G; dg)$, then $\rho(f)$ is a compact operator. Thus G is* $\mathrm{Type}\ I$.

Proof. By the theorem, ρ is TCI and K-finite. $\rho_\kappa(\tau_\kappa)\rho(f) = \rho(\tau_\kappa * f)$ are operators of finite rank. The sum

$$\sum_{\kappa\in\widehat{K}} \rho_\kappa(\tau_\kappa)\rho(f)$$

converges strongly to $\rho(f)$, so $\rho(f)$ is a strong limit of finite rank operators, and therefore compact. □

Remark. The above results guarantee that irreducible ρ which are K-finite will play a central role in the following. The goal is to prove the existence of a distributional character for ρ.

Definition. Let $\{X_1, \dots, X_n\}$ be generators of $\mathfrak{k}$ chosen such that $\langle X_i, X_j\rangle = -\delta_{ij}$. Define the differential operators

$$\omega_K = -(X_1^2 + \cdots + X_n^2),$$
$$E = 1 + \omega_K.$$

These operators are elements of $\mathcal{Z}(\mathfrak{g})$. As such they act as multiples of the identity on irreducible subspaces. For $\kappa \in \widehat{K}$, define $C_\kappa(E)$ to be the unique eigenvalue of E acting on $V(\kappa)$.

6.12. Theorem.

1. $C_\kappa(E) \geq 1$.
2. *For sufficiently large m, $\sum_\kappa C_\kappa(E)^{-m} < \infty$.*
3. *There are constants $c > 0$, $r \geq 0$ such that* $\dim\kappa \leq cC_\kappa(E)^r$ *for all* $\kappa \in \widehat{K}$.

Proof. Without loss we can take a finite cover of K, and write $K = K_1 \times T$ where K_1 is compact and semisimple and T is a torus. Then $\omega_K = \omega_{K_1} + \omega_T$. Clearly $\omega_K \geq 0$ so 1 is obvious.

If we consider ω_T, we see that $\widehat{T} = \mathbb{Z}$, $\omega_T = -d^2/d\theta^2$, $C_n^{(T)} = 1 + n^2$, $\dim n = 1$. Therefore 2 and 3 are clear for the torus part of K.

Now consider the semisimple part K_1; from the basic structure theory of semisimple Lie algebras,

$$\mathfrak{k}_1 = \mathfrak{h} \oplus \mathfrak{k}_{\alpha_1} \oplus \mathfrak{k}_{\alpha_2} \oplus \cdots \oplus \mathfrak{k}_{\alpha_N},$$

where

$$\mathfrak{h} = \sum_{\alpha \in R_\pm(\mathfrak{k}_1)} [\mathfrak{k}_\alpha, \mathfrak{k}_{-\alpha}] .$$

Then

$$\omega_{K_1} = H_1^2 + \cdots + H_M^2 + A_1(X_{\alpha_1}X_{-\alpha_1} + X_{-\alpha_1}X_{\alpha_1}) + A_2(X_{\alpha_2}X_{-\alpha_2} + X_{-\alpha_2}X_{\alpha_2}) + \cdots .$$

By induction it is sufficient to consider $\mathfrak{k}_1 = \mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{su}(2)$. In that case $\widehat{K_1} = \mathbb{Z}^+ = \{0, 1, 2, \dots\}$, $C_n(E) = (n+1)^2$, and $\dim n = n+1$. Then the results 2 and 3 follow.

Finally, given 1,2,and 3 for K_1 and T separately, and using $\omega_K = \omega_{K_1} + \omega_T$, the results follow for K. □

Definition. Let G be connected, semisimple, with finite center. The Harish-Chandra character Θ_ρ is given by

$$\Theta_\rho = \mathrm{Tr}\,(\rho(f)), \quad f \in C_0^\infty(G)$$
$$\rho(f) = \int_G f(g)\rho(g)\, dg.$$

6.13. Theorem. *Let G be connected, semisimple, with finite center. Let (ρ, V) be a unitary irreducible representation of G. Then $\rho(f)$ is trace-class for $f \in C_0^\infty(G)$, so the above definition makes sense.*

Proof. By the K-finiteness result, $\dim V(\kappa) < \infty$. Let $\{e_i^{(k)}\}$ be an orthonormal basis for $V(\kappa)$. Now

$$\left|\left\langle e_i^{(k_1)}, \rho(f)e_i^{(k_2)}\right\rangle\right| = C_{\kappa_1}(E)^{-r}C_{\kappa_2}(E)^{-r}\left|\left\langle e_i^{(k_1)}, E^r\rho(f)E^r e_i^{(k_2)}\right\rangle\right| .$$

Define the operator D^r by

$$\int f(g)E^r\rho(g)E^r = \int (D^r f)(g)\rho(g).$$

If P_κ is the projection onto $V(\kappa)$ then

$$|\int f(g)E^r\rho(g)E^r| \le \left[\sum_\kappa \left\|P_\kappa E^{2r}P_\kappa\right\|\right]\int |f|\, \|\rho(g)\| .$$

The numbers $\|P_\kappa E^{2r}P_\kappa\|$ are simply $C_\kappa(E)^2$, so they decrease to zero sufficiently rapidly for D^r to be a differential operator on f. Then we write

$$|\left\langle e_i^{(k_1)}, \rho(f)e_i^{(k_2)}\right\rangle| \le C_{\kappa_1}(E)^{-r}C_{\kappa_2}(E)^{-r}\int_G \|\rho\|\, |D^r f|.$$

Then, using $\dim V(\kappa) < \infty$ again,

$$\sum_{k_1,k_2,i,j} |\left\langle e_i^{(k_1)}, \rho(f)e_j^{(k_2)}\right\rangle| \le C\left[\sum_{k_1,k_2} C_{\kappa_1}(E)^{-r}C_{\kappa_2}(E)^{-r}\right]\int_G \|\rho\|\, |D^r f|.$$

By the previous lemma, we can choose r sufficiently large that the sum converges, and so

$$\sum_{k_1,k_2,i,j} |\left\langle e_i^{(k_1)}, \rho(f)e_j^{(k_2)}\right\rangle| \leq C' \int_G \|\rho\|\,|D^r f|.$$

The right-hand side is a seminorm on $C_0^\infty(G)$, so

$$\sum_{k_1,k_2,i,j} |\left\langle e_i^{(k_1)}, \rho(f)e_j^{(k_2)}\right\rangle| < \infty.$$

This implies $\rho(f)$ is trace-class. □

6.14. Theorem. *Θ_ρ is a distribution on $C_0^\infty(G)$.*

Proof. $\sum_{n=1} k\,\langle e_n, \rho(f)e_n\rangle$ is continuous on $C_0^\infty(G)$ and tends to $\Theta_\rho(f)$ as $k \to \infty$. $C_0^\infty(G)$ is an inductive limit of Frechet spaces, so $\Theta_\rho(f)$ is continuous on $C_0^\infty(G)$ by the Banach-Steinhaus theorem. □

Remark. The second result of Harish-Chandra, which is the deepest of this set of results, is the fact that $\Theta_\rho(f)$ is actually represented by a locally L^1 function which is moreover “almost” analytic. This hinges on the use of differential equations which follow from the $\mathrm{Ad}\,(G)$ invariance of $\Theta_\rho(f)$. The invariance follows from the following computation

$$\begin{aligned}\Theta_\rho(f\cdot \mathrm{Ad}\,(g')) &= \mathrm{Tr}\left(\int_G f(g'gg'^{-1})\rho(g)\,dg\right)\\ &= \mathrm{Tr}\left(\int_G f(g)\rho(g'gg'^{-1})\Delta_G(g)\,dg\right)\end{aligned}$$

since $\Delta_G = 1$

$$\begin{aligned}&= \mathrm{Tr}\left(\int_G f(g)\rho(g'^{-1}gg')\,dg\right)\\ &= \mathrm{Tr}\left(\rho(g')^{-1}\rho(f)\rho(g')\right)\\ &= \mathrm{Tr}\left(\rho(f)\right)\\ &= \Theta_\rho(f).\end{aligned}$$

Remark. The local statement is, for $z \in \mathcal{Z}\,(\mathfrak{g})$,

$$\begin{aligned}(z\Theta_\rho)(f) &= \Theta_\rho(zf)\\ &= \mathrm{Tr}\left(\rho(zf)\right)\\ &= \mathrm{Tr}\left(\chi_\rho(z)\rho(f)\right)\\ &= \chi_\rho(z)\mathrm{Tr}\left(\rho(f)\right)\\ &= \chi_\rho(z)\Theta_\rho(f).\end{aligned}$$

This is a system of differential equations,

$$z\Theta_\rho = \chi_\rho(z)\Theta_\rho, \quad z \in \mathcal{Z}\,(\mathfrak{g})\,.$$

Definition. The regular set of G is defined by

$$\mathrm{Reg}\,(G) = \left\{ g \in G \,:\, \mathfrak{g}^{\mathrm{Ad}(g)} \text{ is a Cartan subalgebra of } \mathfrak{g} \right\}.$$

6.15. Theorem. *Let Θ be an invariant eigendistribution on G. Then Θ is represented by integration against a locally L^1 function which is analytic on* $\mathrm{Reg}\,(G)$.

Proof. The original Harish-Chandra proof is very complicated. For a simplified version, see [AS77]. □

Remark. A discussion using the example of $\mathrm{SL}_2\,(\mathbb{R})$ is given by Schmid [Sch77]. A discussion of the Harish-Chandra approach is given in some detail by Varadarajan [Var89, p. 208-220].

6.5 Induced Representations of Lie Groups

Remark. The concept of induced representations has been introduced previously for general topological groups. In the case of Lie groups, they have an interpretation in terms of vector bundles over $H\backslash G$.

Remark. For any $x \in H\backslash G$, we can get to any point in a neighbourhood of x by the right action of G. In other words, given $x \in H\backslash G$ and a neighbourhood $W \ni x$, there exists a smooth map $s : W \longrightarrow G$ such that

$$xs(y) = y, \quad \forall y \in W.$$

This is easy to show using the $\exp$ map.

Remark. Cover $H\backslash G$ by nbhds. of the above form, $\{W_i\}$. Define transition mappings by

$$\begin{aligned} s_{ij} : W_i \cap W_j &\longrightarrow G \\ y &\mapsto s_i(y)s_j(y)^{-1}. \end{aligned}$$

Let ρ be a finite-dimensional representation of $H \subset G$ on a space V. Then the covering $\{W_i\}$ and the maps $y \mapsto \rho(s_{ij}(y))$ define a vector bundle over $H\backslash G$ which we denote E_ρ. Clearly E_ρ depends only on the equivalence class of ρ. E_ρ admits a G action, and so it is a G-bundle.

Remark. Given a G-bundle E over $H\backslash G$, the vector space of sections $\Gamma\,(E)$ carries a representation of G,

$$(\rho(g)s)(x) = s(xg) \quad x \in H\backslash G, g \in G, s \in \Gamma\,(E)\,.$$

6.16. Theorem. *Every G-bundle over $H\backslash G$ is equivalent to a bundle E_ρ for some representation ρ.*

Proof. Construct ρ in the following way. Let $x_0 \in H\backslash G$ be an arbitrary point stabilized by H. Then vectors in V_{x_0} remain in V_{x_0} under the right-action by H. Therefore this action defines a representation ρ on V_{x_0}. □

Definition. Let $C^\infty\,(G, H, \rho_H)$ be the space of smooth functions on G with values in V which satisfy

$$F(hg) = \rho_H(h)F(g).$$

$C^\infty\,(G, H, \rho)$ is right-invariant. Therefore we can define a representation of G,

$$(\rho(g)F)(g_1) = F(g_1 g).$$

6.17. Theorem. *The representation on $C^\infty(G, H, \rho_H)$ defined here is equivalent to the representation on $\Gamma(E_{\rho_H})$ given above.*

Proof. Let $s \in \Gamma(E_{\rho_H})$. Define f_s such that

$$(\pi(g), \pi_* f_s(g)) = (Hg, s(g)).$$

The relation $s \leftrightarrow f_s$ is an isomorphism of $\Gamma(E_{\rho_H})$ onto $C^\infty(G, H, \rho_H)$. $\square$

Definition. Suppose now that H is connected. Define a representation σ of H on $C^\infty(G, V)$ by

$$(\sigma(h)f)(g) = \rho_H(h)f(h^{-1}g).$$

Then $C^\infty(G, H, \rho_H) = \{F \in C^\infty(G, V) : F \text{ is invariant under } \sigma\}$. Let σ_* be the corresponding representation of the Lie algebra of H, $\mathfrak{h}$.

6.18. Theorem. *$C^\infty(G, H, \rho_H)$ coincides with the space of solutions of the system of equations*

$$\sigma_*(X)f = 0, \quad X \in \mathfrak{h}, f \in C^\infty(G, V).$$

Proof. Since H is connected, it is generated by an arbitrary neighbourhood of the identity. Some neighbourhood of the identity is covered by $\exp$, though not necessarily all of G. Pick one such generating neighbourhood. Invariance of $C^\infty(G, H, \rho_H)$ by $\sigma(G)$ translates into local invariance of $C^\infty(G, V)$ by σ_* [take the derivative], and this must hold everywhere in G. $\square$

Remark. The equation $\sigma_*(X)f = 0$ is equivalent to

$$\tau_X f + \rho_{H*}(X)f = 0, \quad X \in \mathfrak{h},$$

where τ_X is the left-translation on G corresponding to $X \in \mathfrak{g}$. This invariance condition admits a very interesting construction. Complexify the Lie algebra, $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$. Let $\mathfrak{n}$ be the complex hull of the real subalgebra $\mathfrak{h}$ corresponding to $H \subset G$. Let r be a representation of the complex algebra $\mathfrak{n}$ corresponding to the representation ρ_H of H. Then the solutions of

$$\tau_X f + r(X)f = 0, \quad X \in \mathfrak{n}$$

coincide with the solutions of

$$\tau_X f + \rho_{H*}(X)f = 0, \quad X \in \mathfrak{h}.$$

Remark. As an example, consider $G = \mathbb{R}^2$, $\mathfrak{g} = \mathbb{R}^2$, $\mathfrak{g}_\mathbb{C} = \mathbb{C}^2$. Let $\mathfrak{n}$ be the one-complex-dimensional subalgebra generated by $X + iY$. Let r be the one-complex-dimensional representation of $\mathfrak{n}$ which takes the value $\lambda \in \mathbb{C}$ at $X + iY$. Then the invariance equation is

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} + \lambda f = 0.$$

Therefore $f = \exp(-\lambda\bar{z}/2)\phi(z)$, with ϕ holomorphic. Therefore there is a certain built-in holomorphicity; the invariance equations are related to conditions of holomorphicity. See [Kir76, p. 202].

6.6 Principal Series

Remark. In the following let $G = KAN$ be an Iwasawa decomposition of G, where G is connected and semisimple. Let $P = MAN$ be a minimal parabolic subgroup.

6.19. Theorem. *Every finite-dimensional irreducible unitary representation ρ_P of P has the form*

$$\rho_P(man) = \chi(a)\rho_M(m),$$

where χ is a character of A and ρ_M is an irreducible representation of M.

Proof. Recall that AN is connected and solvable. Therefore all its irreducible representations are one-dimensional. So there is a character of AN and a vector v_0 such that $\rho_P(an)v_0 = \chi(an)v_0$. But $\chi(n) = 1$ for all $n \in N$. $\rho_P(M)v_0$ is stable under $\rho_P(man)$. But $\rho_P(man)$ is irreducible by assumption. Therefore $\rho_P(M)v_0 = \mathcal{H}$. Therefore $\rho_P(an)v = \chi(a)v$ for all $v \in \mathcal{H}$. $\square$

Definition. Let $\tilde{M}$ be the normalizer of A in K. Then $W_r = \tilde{M}/M$ is called the restricted Weyl group of G. W_r acts on representations of MA.

Definition. Let $\mathfrak{a}$ be the Lie algebra of A. Let

$$\delta = \frac{1}{2} \sum_{\alpha \in \mathfrak{a}^*} \dim \mathfrak{g}_\alpha.$$

Define $\beta_{\mu,\eta}(man) = \mu(m) \exp((\delta + i\eta) \log(a))$ for $\eta \in \mathfrak{a}'$, $\mu \in \widehat{M}$. This is an irreducible unitary representation of P on $V(\mu)$.

Definition. Consider the induced representations

$$T(\mu, \eta) = \mathrm{Ind}\left(G, P, \beta_{\mu,\eta}\right).$$

These representations form the principal series of G.

6.20. Theorem.

- *$T(\mu, \eta)$ is unitary if and only if η is real, $\eta : \mathfrak{a} \longrightarrow \mathbb{R}$.*
- *$T(1, \eta)$ is irreducible for all $\eta \in \mathfrak{a}'$.*

Proof. [Bru56] [P+67] [Kos69] [Wal71] [KS71]. $\square$

6.7 Discrete Series

Remark. Recall the definition of the discrete series of representations of a locally compact group G. For the case when G is a semisimple Lie group, a great deal can be said about the construction of its discrete series. These are the celebrated results of Harish-Chandra, and they are necessary to proceed with harmonic analysis on the group. In fact, the discrete series and the principal series together have full Plancherel measure, so no other representations are required for harmonic analysis.

6.21. Theorem. *T.F.A.E*

- $\widehat{G}_{\mathrm{disc}} \neq \emptyset$
- $\mathrm{rank}\,(G) = \mathrm{rank}\,(K)$
- G *has a compact Cartan subgroup.*

Proof. [HC66]. □

Remark. The original discrete series construction of Harish-Chandra is long and difficult. Atiyah and Schmid have given a simplified approach based on L^2 index theory for the Dirac operator on an associated compact quotient space [AS77].

Remark. See [Wol80, p. 123].

Chapter 7

Co-Adjoint Orbits

Remark. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Denote the dual space of $\mathfrak{g}$ by $\mathfrak{g}^*$. Recall that $\mathfrak{g}$ carries a representation of G via the adjoint map $\mathrm{Ad}\,() : G \longrightarrow \mathrm{Aut}\,(\mathfrak{g})$, which is the derivative of the conjugation map $\mathrm{Ad}\,() = A_*(1)$, $A(g) : h \mapsto ghg^{-1}$. The space $\mathfrak{g}^*$ is naturally realized as the space of left-invariant differential 1-forms on G.

Definition. The coadjoint representation of G acts in $\mathfrak{g}^*$ by right translations.

Remark. Consider the case of matrix groups, where $TG_p \cong \mathrm{Mat}_n(\mathbb{C})$.

7.1. Theorem. *Let $G = \mathrm{GL}_n\,(\mathbb{C})$. A left-invariant vector field on G is a matrix-valued function $v(g)$. Every left-invariant vector field on G has the form*

$$v_A(g) = gA, \quad \textit{for some } A \in \mathrm{Mat}_n(\mathbb{C}).$$

Under the action of right translation we have

$$(r_{y*})v_A = v_{y^{-1}Ay}, \quad y \in G.$$

Proof. Left and right translation are linear maps. Therefore they are equal to their derivatives. So we have

$$\tau_{y*}v(g) = yv(y^{-1}g) \quad \text{left action} \qquad r_{y*}v(g) = v(gy^{-1})y \quad \text{right action}.$$

Left-invariance means $v(g) = yv(y^{-1}g)$, therefore $y^{-1}v(g) = v(y^{-1}g)$. Let $A = v(1)$, so $v(y^{-1}) = y^{-1}A$. Thus $v(g) = gA$ for some $A \in \mathrm{Mat}_n(\mathbb{C})$. Finally, given $v(g) = gA$, the action of right translation is clearly as claimed. $\square$

7.2. Theorem. *Let $G = \mathrm{GL}_n\,(\mathbb{C})$. A left-invariant 1-form on G is a matrix-valued function $\omega(g)$. Every left-invariant 1-form on G has the form*

$$\omega_B(g) = Bg^{-1}, \quad \textit{for some } B \in \mathrm{Mat}_n(\mathbb{C}).$$

Under right translation we have

$$(r_y^*)\omega_B = \omega_{yBy^{-1}}, \quad y \in G.$$

Proof. We identify $\mathrm{Mat}_n(\mathbb{C})^* \cong \mathrm{Mat}_n(\mathbb{C})$ using the dual pairing over $\mathbb{R}$ $\langle X, Y\rangle = \mathrm{Re}\,\mathrm{Tr}\,(XY)$.

The action of r_y^* on forms is defined by the dual pairing and the action of r_{y*} on vector fields. We write

$$\begin{aligned} \mathrm{Tr}\,\big((r_y^*\omega)v\big) &= \mathrm{Tr}\,(\omega(r_{y*}v)) \\ &= \mathrm{Tr}\,(\omega v y) \\ &= \mathrm{Tr}\,(y\omega v)\,. \end{aligned}$$

This must be true for arbitrary v, so we have $r_y^*\omega = y\omega(gy)$. Similarly $\tau_y^*\omega = \omega(yg)y$. Left-invariance of ω means $\omega(g) = \omega(yg)y$. Let $B = \omega(1)$, so $\omega(y) = By^{-1}$. □

Remark. Note that orbits of G in $\mathfrak{g}^*$ under this coadjoint representation represent classes of similar matrices.

Remark. In our discussion of induced representations for Lie groups we encountered G-bundles over $H\backslash G$, which could be written in terms of vector-valued functions. As a special case we note the following facts.

G-invariant differential forms on $H\backslash G$ correspond uniquely to H-invariant elements of $\Lambda\left(\mathfrak{h}^\perp\right)$. A function on G corresponding to a G-invariant form is a constant with values in $\Lambda\left(\mathfrak{h}^\perp\right)$, and its value is an H-invariant element.

If Φ is an invariant k-form on $H\backslash G$ which corresponds to the exterior form $\phi \in \Lambda^k\left(\mathfrak{h}^\perp\right)$, then $d\Phi$ corresponds to

$$\Lambda^{k+1}\left(\mathfrak{h}^\perp\right) \ni d\phi(X_1, \ldots, X_{k+1}) = \frac{1}{k+1}\sum_{i<j}(-1)^{i+j+1}\phi([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots).$$

7.3. Theorem. *Consider the orbits of G in the coadjoint representation. Let Ω be such a coadjoint orbit. Let $F \in \Omega$, an arbitrary point on the orbit. Let G_F be the stabilizer of F. Then we have*

$$\mathfrak{g}_F = \ker\,(B_F)$$

where

$$B_F(X, Y) = \langle F, [X, Y]\rangle\,.$$

Proof. $\ker\,(B_F) = \{X \in \mathfrak{g} \,:\, B_F(X, Y) = 0\,\forall Y \in \mathfrak{g}\}$. Now

$$B_F(X, Y) = \left\langle \frac{d}{dt}K(\exp tX)F|_{t=0}, Y\right\rangle,$$

where $K(g)$ is the coadjoint representation. The assertion follows. □

Remark. Now we will construct a 2-form on the space of a given orbit Ω. $B_F(X, Y)$ depends only on $p(X), p(Y)$ where $p : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{g}_F$ is the natural projection. There B_F gives rise to a skew-symmetric bilinear form on $\mathfrak{g}/\mathfrak{g}_F$. Clearly this form, which we call $\tilde{B}_F$, is non-degenerate.

7.4. Theorem. *$\tilde{B}_F$ is a G_F-invariant element of $\Lambda^2\left(\mathfrak{g}/\mathfrak{g}_F\right)^*$.*

Proof.

$$\begin{aligned}\rho(g)\tilde{B}_F(X,Y) &= \tilde{B}_F(\operatorname{Ad}\left(g^{-1}\right)X, \operatorname{Ad}\left(g^{-1}\right)Y)\\ &= \langle K(g)F, [X,Y]\rangle\\ &= 0.\end{aligned}$$

□

Remark. Using the correspondence between elements of $\Lambda\left(\mathfrak{h}^{\perp}\right)$ and G-invariant forms, we have a non-degenerate G-invariant 2-form B_Ω on $\Omega = G_F\backslash G$. It can be shown that B_Ω does not depend on the choice of $F \in \Omega$. Using the formula for $d\Phi$ given above, it can be shown that B_Ω is closed.

Therefore we have found a G-invariant symplectic form on the orbit space Ω; Ω is a homogeneous symplectic space.

7.5. Theorem. *Let G be a connected Lie group. Then every G-symplectic manifold is locally isomorphic [i.e up to taking covering spaces] to a coadjoint orbit of G or of a central extension of G with the aid of $\mathbb{R}$.*

Proof. See [Kir76, p. 234]. □

Definition. An orbit is called integral if B_Ω belongs to an integer cohomology class, i.e. the integral of B_Ω over an arbitrary 2-cycle is an integer.

7.6. Theorem (Borel-Weil-Bott). *All irreducible representations of a compact, connected, simply connected Lie group G correspond to integral G-orbits of maximal dimension in $\mathfrak{g}^*$.*

Proof. See [Kir76, p. 241]. □

Remark. The condition of integrality is equivalent to the quantization condition of "old quantum theory",

$$\oint pdq = nh.$$

Definition. Define a generalized function I_Ω by

$$\langle I_\Omega, \phi\rangle = \int_\Omega \left[\int_U \phi(\exp X)e^{2\pi i\langle F,X\rangle}dX\right]d\beta_\Omega(F),$$

where $\phi \in C_0^\infty(G)$ is defined in an open region $V \in G$, and U is the inverse image of V in $\mathfrak{g}$; dX is ordinary Lebesgue measure on $\mathfrak{g}^*$, and

$$d\beta_\Omega(F) = \frac{1}{k!}B_\Omega \wedge \cdots \wedge B_\Omega, \quad 2k \text{ factors.}$$

Remark. One conjectures that the character of a representation associated to an orbit is given by

$$\chi_\Omega = p_\Omega^{-1} I_\Omega,$$

where p_Ω is some function in $C^\infty(G)$ which is invariant under inner automorphisms, equals 1 at the identity, and is different from zero on the open region V. This is the so-called universal character formula. So far this conjecture has been proven in the following cases.

- Representations of compact simply connected Lie G.
- Representations of exponential groups.
- Representations of $\mathrm{GL}_2(\mathbb{R})$.
- Representations of the principal series of noncompact semi-simple groups.

See the papers of Kirillov.

Remark. For representations corresponding to orbits of maximal dimension, a single universal function p_Ω can be selected,

$$q(\exp X) = \det(\zeta(\mathrm{ad}(X)))$$
$$\zeta(t) = \frac{\sinh(t/2)}{t/2}.$$

Chapter 8

Projective Representations

Remark. This material is taken from Kirillov's book. See [Kir76].

Definition. Let V be an n-dimensional linear space over a field K. Let PV be the corresponding projective space. The group of automorphisms of PV is

$$\mathrm{Aut}\,(PV) \cong \mathrm{GL}_n\,(K)/\{K \cdot 1\} \equiv P\mathrm{GL}_n\,(K)\,.$$

Similarly, if $\mathcal{H}$ is a Hilbert space, we write $P\mathcal{H}$ for its associated projective space. Let $P\tilde{U}(\mathcal{H})$ be the group of isometries of $P\mathcal{H}$. Then

$$P\tilde{U}(\mathcal{H}) \cong \tilde{U}(\mathcal{H})/\{\lambda \cdot 1\},$$

where $\tilde{U}\mathcal{H}$ is the group of operators which are products of unitary operators together with complex conjugation [unitary and anti-unitary operators].

8.1. Theorem. *Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of bounded operators on $\mathcal{H}$. Then*

$$\mathrm{Aut}\,(\mathcal{B}(\mathcal{H})) \cong P\tilde{U}(\mathcal{H}).$$

8.2. Theorem. *The connected component of the identity in $P\tilde{U}(\mathcal{H})$ is*

$$PU(\mathcal{H}) \equiv U(\mathcal{H})/\{\lambda \cdot 1\}.$$

Definition. A projective representation of G on a finite-dimensional space is a homomorphism of G into $P\mathrm{GL}_n\,(K)$. So every projective representation of G on an n-dimensional projective space is a map

$$t : G \longrightarrow \mathrm{GL}_n\,(K)$$

such that

$$t(g_1)t(g_2) = c(g_1, g_2)t(g_1g_2)$$
$$c : G \times G \longrightarrow K\setminus\{0\}\,.$$

By equating $t(g_1g_2)t(g_3)$ and $t(g_1)t(g_2g_3)$ we require

$$c(g_1, g_2)c(g_1g_2, g_3) = c(g_1, g_2g_3)c(g_2, g_3).$$

Therefore a projective representation uniquely defines an element of $\mathrm{H}^2\,(G, K\setminus\{0\})$.

Definition. A projective unitary representation of G is a homomorphism of G into $P\tilde{U}(\mathcal{H})$. If G is connected then any such representation is a homomorphism into $PU(\mathcal{H})$.

Remark. Let G_0 be a commutative subgroup of G. Consider the central extension of G by G_0,

$$1 \to G_0 \to \tilde{G} \to G \to 1.$$

Clearly we can obtain projective representations of G from linear representations of $\tilde{G}$, $\tilde{\rho}$, by mapping $g \mapsto \tilde{\rho}(\tilde{g})$, with $\tilde{g}$ an arbitrary inverse image of π. We only require that $\tilde{\rho}(g_0)$ is a scalar operator for all $g_0 \in G_0$.

8.3. Theorem. *Every projective representation of G is obtained from an extension by some group $\tilde{G}$.*

Proof. Projective representations correspond to elements of $\mathrm{H}^2\left(G, K\backslash\{0\}\right)$. as we have already noted. Let $\tilde{\rho}$ be as described above, i.e. a linear representation such that $\tilde{\rho}(g_0)$ is scalar. A central extension is then possible, identifying G_0 with a subgroup of $K\backslash\{0\}$. The classes of such extensions are as well classified by $\mathrm{H}^2\left(G, K\backslash\{0\}\right)$. □

Definition. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We can define an algebra cohomology as follows. Let $Z^2(\mathfrak{g}, \mathbb{R})$ be the set of functions $c : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ with the properties

1. c is bilinear and skew-symmetric.
2. $c([X_1, X_2], X_3) + c([X_2, X_3], X_1) + c([X_3, X_1], X_2) = 0.$

Let $B^2(\mathfrak{g}, \mathbb{R})$ be the subspace of $Z^2(\mathfrak{g}, \mathbb{R})$ consisting of functions of the form

$$c(X_1, X_2) = \langle F, [X_1, X_2]\rangle, \quad F \in \mathfrak{g}^*.$$

Define a cohomology group

$$\mathrm{H}^2\left(\mathfrak{g}, \mathbb{R}\right) = Z^2(\mathfrak{g}, \mathbb{R})/B^2(\mathfrak{g}, \mathbb{R}).$$

8.4. Theorem. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1$ is semisimple and $\mathfrak{g}_2$ is a solvable ideal. Then $\mathrm{H}^2\left(\mathfrak{g}, \mathbb{R}\right)$ is isomorphic to the subspace of $\mathrm{H}^2\left(\mathfrak{g}_2, \mathbb{R}\right)$ generated by $c \in \mathrm{H}^2\left(\mathfrak{g}_2, \mathbb{R}\right)$ satisfying*

$$c([X_1, Y], X_2) = c(X_1, [Y, X_2]), \quad X_1, X_2 \in \mathfrak{g}_2, Y \in \mathfrak{g}_1.$$

8.5. Corollary. *Every projective representation of a connected and simply connected semisimple Lie group is obtained from a linear representation of the group.*

Chapter 9

Analysis on Coset Spaces

9.1 Differential Operators

Definition. Let G be a Lie group. Let $D(G)$ denote the set of left-invariant differential operators on G. Given $X \in \mathfrak{g}$ define a vector field $\tilde{X}$ on G by

$$\begin{aligned}(\tilde{X}f)(g) &= X(f \circ l_g) \\ &= \frac{d}{dt} f(g \exp tX)|_{t=0}.\end{aligned}$$

Then $(\tilde{X}(f \circ l_h)(h^{-1}g) = (\tilde{X}f)(g)$. So $\tilde{X} \in D(G)$.

9.1. Theorem. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then there exists a unique linear bijection*

$$\lambda : \mathrm{Sym}\,\mathfrak{g} \longrightarrow D(G),$$

such that $\lambda(X^m) = \tilde{X}^m$, where $\mathrm{Sym}\,\mathfrak{g}$ is the symmetric algebra over the vector space $\mathfrak{g}$. If $\{X_i\}$ is a basis of $\mathfrak{g}$ and $P \in \mathrm{Sym}\,\mathfrak{g}$ then

$$(\lambda(P)f)(g) = P(\partial_1, \ldots, \partial_n)f(g \exp(t_i X_i))|_{t=0}.$$

Proof. This is not too difficult. See [Hel84, p. 281]. $\square$

9.2. Theorem. *Suppose G is connected. Let $\mathcal{Z}(D)$ be the center of $D(G)$. Let $I(\mathfrak{g}) \subset \mathrm{Sym}\,\mathfrak{g}$ be the set of $\mathrm{Ad}\,(G)$ invariants. Then*

$$\lambda(I(\mathfrak{g})) = \mathcal{Z}(D),$$

and $\mathcal{Z}(D)$ is precisely the set of bi-invariant differential operators on G.

Proof. It is easy to see that $\tilde{X}D = D\tilde{X}$ if and only if

$$D^{r_{\exp tX}} = D$$

for all $t \in \mathbb{R}$. Also $\lambda(\mathrm{Ad}\,(g)\,P) = \mathrm{Ad}\,(g)\,\lambda(P)$. So both statements are proven. $\square$

Definition. Let H be a closed subgroup of G, with Lie algebra $\mathfrak{h}$. Let $\mathfrak{m}$ be any complementary subspace, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Use $\mathfrak{m}$ and $\exp(\cdot)$ to coordinatize G/H.

Define G/H to be reductive if $\mathfrak{m}$ can be chosen such that $\mathrm{Ad}_G(h)\mathfrak{m} \subset \mathfrak{m}$, $h \in H$. In particular, if $\mathrm{Ad}_G(H)$ is compact then G/H is reductive; if H is compact then clearly G/H is reductive.

Definition. Define $D_H(G) = \{D \in D(G) : D^{r_h} = D$ for all $h \in H\}$, i.e. the set of H-invariant operators on G. If f is a function on G/H, define $\tilde{f} = f \circ \pi$.

Definition. Define $D(G/H)$ to be the set of differential operators on G/H which are G-left-invariant, $D^{l_g} = D$.

9.3. Theorem. *Assume G/H is reductive. Define $\mu : u \mapsto D_u$,*

$$\widetilde{D_u f} = u\tilde{f},$$

for f a function on G/H. Then μ is a homomorphism of $D_H(G)$ onto $D(G/H)$ with $\ker(\mu) = D_H(G) \cap D(G)\mathfrak{h}$. So

$$D(G/H) \cong D_H(G)/\ker(\mu).$$

Proof. See [Hel84, p. 285]. □

Definition. A homogeneous space is called two-point homogeneous if for any two pairs of points (p, q), (p', q'), with $d(p, q) = d(p', q')$, there exists an isometry g with $g(p) = p'$, $g(q) = q'$.

Remark. It can be shown that a Riemannian manifold is a two-point homogeneous space if and only if it is isotropic.

9.4. Theorem. *Let G/K be two-point homogeneous. Then $D(G/H)$ consists of polynomials in the Laplace-Beltrami operator.*

Proof. By the property of two-point homogeneity, $\mathrm{Ad}_G K$ acts transitively on the unit sphere in any tangent space of G/K. Therefore, the set of $\mathrm{Ad}_G K$ invariants in $\mathrm{Sym}\,\mathfrak{m}$ is generated by $X_1^2 + \cdots + X_r^2$ where $\{X_i\}$ is an orthonormal basis of $\mathfrak{m}$. □

9.5. Theorem. *Let G/K be a symmetric space with K compact. Then $D(G/K)$ is commutative.*

Proof. See [Hel84, p. 293]. □

9.2 Spherical Functions

Definition. Let G be a connected Lie group with K a compact subgroup. Let $\phi : G/K \longrightarrow \mathbb{C}$ be a smooth function with $\phi(\pi(1)) = 1$. Then ϕ is called a spherical function if the following hold.

- $\phi^{l_k} = \phi$ for all $k \in K$.
- $D\phi = \lambda_D \phi$ for all $D \in D(G/K)$; $\lambda_D \in \mathbb{C}$.

Definition. Recall the definition $\tilde{\phi} = \phi \circ \pi$. We say that $\tilde{\phi}$ is spherical on G if ϕ is spherical on G/K. If $\tilde{\phi}$ is spherical on G then it is K-bi-invariant,

$$\tilde{\phi}(kgk') = \tilde{\phi}(g), \text{ for all } g \in G, k, k' \in K.$$

9.6. Theorem. *Let $\phi : G \longrightarrow \mathbb{C}$ be continuous and not identically zero. Then ϕ is spherical if and only if*

$$\int_K \phi(xky)dk = \phi(x)\phi(y).$$

Proof. See [Hel84, p. 400]. ☐

Chapter 10

Harmonic Analysis I

10.1 Introduction

Recall our basic goals given a topological group G.

- Find $\widehat{G}$.
- Find the Plancherel measure on $\widehat{G}$.
- Find an analogue of the Weyl character formula, i.e. determine the characters.

These results form the foundation for harmonic analysis on any group G.

10.2 Classical Fourier Series

Remark. The theory of classical Fourier series deals with functions on the torus T^n. T^n is a compact abelian group and its irreducible (and thus one-dimensional) unitary representations are precisely the characters

$$\chi_{\vec{m}}(\vec{\theta}) = \exp\left[2\pi i \vec{m} \cdot \vec{\theta}\right], \quad \vec{m} \in \mathbb{Z}^n.$$

where we have realized T^n as $[0,1]^n$.

Definition. For $f \in L^2(T^n; d\theta)$, the Fourier transform of f is given by

$$(\mathcal{F}f)(\vec{m}) \equiv \widehat{f}(\vec{m}) = \int d^n\theta\, f(\vec{\theta})\overline{\chi_{\vec{m}}(\vec{\theta})}.$$

Remark. The Fourier transform has the interesting property that it diagonalizes the regular representation, $(r(\vec{\phi})f)(\vec{\theta}) = f(-\vec{\phi} + \vec{\theta})$, in the sense that the representation $\mathcal{F} \circ r \circ \mathcal{F}^{-1}$ on $L^2(\mathbb{Z}^n; \delta)$ is diagonal in the standard basis of $L^2(\mathbb{Z}^n; \delta)$.

Remark. When f is smooth we have the Plancherel formula

$$f(\vec{\theta}) = \sum_{\vec{m}} \widehat{f}(\vec{m}) \chi_{\vec{m}}(\vec{\theta})$$
$$\Longrightarrow f(0) = \sum_{\vec{m}} \widehat{f}(\vec{m}),$$

where the convergence is absolute. This is just the inverse of $\mathcal{F}$.

Remark. The fact that $\mathcal{F}$ is unitary is expressed in the Parseval relation

$$\int_{T^n} f \overline{g} d^n \theta = \sum_{\vec{m}} \widehat{f}(\vec{m}) \overline{\widehat{g}(\vec{m})}.$$

10.3 Classical Fourier Analysis

Remark. Classical Fourier analysis deals with functions on $\mathbb{R}^n$, which is a locally compact abelian group. The Fourier transform is defined first by a map on Schwarz space $\mathcal{S}(\mathbb{R}^n)$,

$$(\mathcal{F}f)(\vec{k}) \equiv \widehat{f}(\vec{k}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \, d^n x.$$

This defines a bounded linear transformation and thus extends to $L^p(\mathbb{R}^n; dx)$.

The Parseval relation on $L^2(\mathbb{R}^n; dx)$ is

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f \overline{g} d^n x = \int_{\mathbb{R}^n} \widehat{f} \overline{\widehat{g}} \, d^n k.$$

When f is smooth the Plancherel formula holds

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(k) e^{i\vec{k}\cdot\vec{x}} d^n k.$$

Remark. If T is a distribution, $T \in \mathcal{S}'(\mathbb{R}^n)$, then its Fourier transform is defined by $\widehat{T}(f) = T(\widehat{f})$, $f \in \mathcal{S}(\mathbb{R}^n)$. A particularly interesting case is $T = \sum_{\vec{n} \in \mathbb{Z}^n} \delta(\vec{x} - \vec{n})$. Then $\widehat{T} = \sum_{\vec{m} \in \mathbb{Z}^n} \delta(\vec{k} - 2\pi\vec{m})$. This is equivalent to the Poisson formula

$$\sum_{\vec{n} \in \mathbb{Z}^n} f(\vec{n}) = \sum_{\vec{q} \in 2\pi\mathbb{Z}^n} \widehat{f}(\vec{q}), \quad f \in \mathcal{S}(\mathbb{R}^n).$$

10.4 Locally Compact Abelian Groups

Remark. The classical theory extends to general locally compact abelian groups. Let G be such a group in the following.

Definition. Recall that the set of characters $\widehat{G}$ is itself a locally compact abelian group when G is ableian. Let $\chi \in \widehat{G}$. Define the Fourier transform of $f \in L^1(G; dg)$ by

$$\widehat{f}(\chi) = \int f(g) \overline{\chi(g)} \, dg.$$

Remark. A small technical effort is required in order to define the Fourier transform on $L^2(G; dg)$. Define a norm

$$\|f\|_0 = \max(\|f\|_\infty, \|f\|_2, \left\|\widehat{f}\right\|_\infty), \quad f \in C_0(G).$$

Define $\mathcal{D}(G)$ to be the completion of $C_0(G)$ with $\|\cdot\|_0$. Convolution and conjugation extend to $\mathcal{D}(G)$ making it a Banach *-algebra, and the Fourier transform $\mathcal{F}$ extends to a *-isomorphism of $\mathcal{D}(G)$ onto a subalgebra of $C(\widehat{G})$ consisting of functions vanishing at infinity; the functions vanishing at infinity are $\|\cdot\|_0$-dense in that subalgebra.

In sketch we do the following. Define a linear functional on $\widehat{\mathcal{D}(G)}$, which is the image of $\mathcal{D}(G)$ under $\mathcal{F}$,

$$\Lambda(\widehat{f}) \equiv f(1), \quad f \in \mathcal{D}(G).$$

Then $\Lambda|_{C_0(\widehat{G})}$ is the Haar integral on $\widehat{G}$ which we can use in the following way,

$$f(g) = (r_{g^{-1}}f)(1) = \int_{\widehat{G}} \widehat{(r_{g^{-1}}f)}(\chi)\, d\chi = \int_{\widehat{G}} \widehat{f}(\chi)\chi(g)\, d\chi.$$

Now $\|f\|_2^2 = (f^* \circ f)(1) = \Lambda(|\widehat{f}|^2)$, and clearly we have $\left\|\widehat{f}\right\|_2 \le \|f\|_2$. But the set $\left\{f \in \mathcal{D}(G) \,:\, \widehat{f} \in C_0(G)\right\}$ is easily seen to be dense in $\mathcal{D}(G)$ with the $\|\cdot\|_2$ norm. This shows that $\left\|\widehat{f}\right\|_2 = \|f\|_2$. Therefore we have the following theorem.

10.1. Theorem (Parseval). *$f \mapsto \widehat{f}$ is a unitary isomorphism of $L^2(G; dg)$ onto $L^2\left(\widehat{G}; d\chi\right)$, for a certain Haar measure $d\chi$ on $\widehat{G}$.*

Definition. Let H be a closed subgroup of an abelian group G. Define the annihilator of H, $\breve{H}$, to be the subgroup

$$\breve{H} \le \widehat{G}$$
$$\breve{H} = \left\{\chi \in \widehat{G} \,:\, \chi(h) = 1,\, h \in H\right\}.$$

Then we have $\widehat{G/H} \cong \breve{H}$ and $\widehat{H} \cong \widehat{G}/\breve{H}$.

Definition. If H is discrete and G/H is compact, then H is called a lattice in G. In that case $\breve{H}$ is also a lattice in $\widehat{G}$.

10.2. Theorem (Poisson Formula). *Let Γ be a lattice in G. Normalize Haar measure so that $\mu(G/\Gamma) = 1$. Suppose that $f \in C(G) \cap L^1(G; dg)$, and $\widehat{f} \in L^1\left(\widehat{G}; d\chi\right)$, and that*

$$\sum_{\gamma \in \Gamma} f(g+\gamma), \quad \sum_{\breve{\gamma} \in \breve{\Gamma}} \widehat{f}(\chi + \breve{\gamma})$$

are uniformly convergent for g, χ varying over compact subsets of G and $\widehat{G}$ respectively. Then we have

$$\sum_{\gamma \in \Gamma} f(\gamma) = \sum_{\breve{\gamma} \in \breve{\Gamma}} \widehat{f}(\breve{\gamma})$$

10.5 Compact Groups

Remark. The compact case is covered completely by the Peter-Weyl theory. This asserts that the matrix elements of the irreducible representations of G are complete and orthonormal in $L^2(G; dg)$. We have seen already how this occurs, in the general context of topological groups.

Remark. The first really nontrivial example is probably $\mathrm{SO}(3)$. Harmonic analysis on $\mathrm{SO}(3)$ is based on application of Peter-Weyl theory and a specific realization of the regular representation. For each $n = 0, 1, 2, \ldots$ define a matrix $A_n \in \mathrm{Mat}(2n+1, 2n+1)$ by

$$Y_l^m(gx) = \sum_{|k| \leq l} (A_l(g))_{l,k} Y_l^k(x), \quad g \in \mathrm{SO}(3).$$

In other words, $A_n(g)$ implements the regular representation of $\mathrm{SO}(3)$ on the representation of dimension $2l+1$. As such, the matrix elements of A_l for all l are complete in $L^2(\mathrm{SO}(3); dg)$; the A_l are a complete set of irreducible unitary representations. [Actually this is more subtle; it apparently fails for $\mathrm{SO}(n)$, $n \geq 4$].

Then we define

$$\widehat{f}(l) = \int_{\mathrm{SO}(3)} f(g) \, [A_l(g)]^\dagger \, dg$$

and

$$f(g) = \sum_{l=0}^{\infty} (2l+1) \mathrm{Tr} \left(\widehat{f}(l) A_l(g) \right).$$

We will encounter more examples later, when considering the group theoretic approach to special functions.

Definition. Let G be a compact group. Let $f \in L^1(G; dg)$. Then the Fourier transform of f is defined to be the operator-valued function on $\widehat{G}$ given by

$$\widehat{f}(\kappa) = \int_G f(g) \rho_\kappa(g^{-1}) dg.$$

Of course $\widehat{f}(\kappa)$ is a finite-dimensional operator, so there are no function-analytic complications.

Remark. The Peter-Weyl theorem shows that, if $f \in L^1(G; dg) \cap L^2(G; dg)$, then

$$\int_G |f(g)|^2 = \sum_{\kappa \in \widehat{G}} \dim \kappa \; \mathrm{Tr} \left(\widehat{f}(\kappa)^* \widehat{f}(\kappa) \right).$$

10.6 Noncompact Groups

Definition. Let G be a locally compact group. Let $f \in L^1(G; dg)$. Define the Fourier transform to be the operator-valued function on $\widehat{G}$ given by

$$\widehat{f}(\kappa) = \int_G f(g) \rho_\kappa(g^{-1}) dg.$$

Chapter 11

Harmonic Analysis II

11.1 Sampling Theorem on the Circle

Definition. Let $f \in C([0,1])$ with a Fourier series repesentation $f(t) = \sum_{m=-\infty}^{\infty} f_m \exp(2\pi i m t)$. We say that f is band-limited with band limit M if $f(t) = \sum_{|m|\leq M} f_m \exp(2\pi i m t)$.

Remark. Band-limited functions on the circle are clearly smooth and periodic, and the finite sum obviously exists pointwise everywhere.

Remark. Because a band-limited function is specified by a finite number of parameters, one imagines that it is possible to reconstruct the function from a finite amount of data, such as the values at some finite set of points. This is true in general, and we can construct many explicit such sampling schemes, of which we will examine the most common shortly.

11.1. Theorem. *Let $\{f_i\}$ be a set of sample values, supposed given at a set of distinct points $\{t_i \in [0,1]\}$, $i = 1, \ldots, 2M+1$. Then amongst the band-limited functions with band limit M there exists a unique function $f(t)$ with $f(t_i) = f_i$.*

Proof. This is an elementary statement. Let $z(t) = \exp(2\pi i t)$. Then a band-limited function with band-limit M is a Laurent polynomial in z, of order M, $f(t) = \sum_{|m|\leq M} f_m z^m$. As such it has a unique analytic continuation to an annulus containing the unit circle and separated from the origin and infinity. Such a Laurent polynomial is uniquely specified by its values at any set of distinct $2M+1$ points within the annulus, in particular by the $2M+1$ values at $z(t_i)$. $\square$

11.2. Theorem. *Let f be a band-limited function on $[0,1]$, with band limit M. Let $t_n = n/(2M+1)$. Then the following equality holds pointwise.*

$$f(t) = \frac{1}{2M+1} \sum_{|n|\leq M} f\left(t_n \mod 1\right) \frac{\sin((2M+1)\pi(t-t_n))}{\sin(\pi(t-t_n))}.$$

Proof. This will be an easy consequence of a more powerful result which we will prove below. $\square$

11.2 Sampling Theorem on the Line

Definition. Let $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and let $\widehat{f}$ be the Fourier transform of f. We say that f is band-limited with band limit W if $\widehat{f}$ is zero outside of $[-W, W]$.

11.3. Theorem. *Let $f \in L^2(\mathbb{R})$ be band-limited with band limit W. Then the following holds,*

$$f(t) = \lim_{N->\infty} \sum_{|n|\leq N} f\left(\frac{n}{2W}\right) S(t - n/2W),$$

with

$$S(t) = \frac{\sin(2\pi W t)}{2\pi W t},$$

where the sum converges absolutely.

Proof. This will be an easy consequence of a more powerful result which we will prove below. □

Remark. The sampling kernel which appears here is clearly related to the one for sampling on the interval given above. One can construct the kernel for the interval by summing the kernel for the line in order to make it periodic, as it must be. This is a sum over equally spaced poles, and the residues imply that the sum is equal to $1/\sin(2\pi t)$ (the theorem of Mittag-Lefler applies). The converse arises by contracting the formula for the circle to the tangent space at zero, a kind of Wigner contraction.

11.3 Sampling as Spectral Analysis on the Dual Space

Remark. Each of the above sampling theorems is a consequence of a more general picture. This general picture shows that the convergence of sampling theory can be interpreted as convergence of expansions in the dual or frequency space.

11.4. Lemma. *Fix a topological space $\mathcal{T}$ and a measure space Ω. Let $f : T \longrightarrow \mathbb{C}$ and $g \in L^2(\Omega)$, and suppose*

$$f(t) = \int_\Omega d\mu(\omega)\, K(t, \omega) g(\omega),$$

where $K(t, \omega)$ is an integral kernel, continuous in each variable, and square-integrable in ω for all t. Let $\{\phi_n(\omega)\}$ be an orthonormal basis for $L^2(\Omega)$. Define $S_n(t) = (K(t, \cdot), \phi_n(\cdot)) = \int d\omega\, K(t, \omega)\phi_n^*(\omega)$. Suppose finally that there exists a set $\{t_n \in \mathcal{T}\}$, such that $K(t_n, \omega) = \phi_n(\omega)$. Define $f_N(t) = \sum_{n\leq N} f(t_n) S_n(t)$. Then

$$f(t) = \lim_{N\to\infty} f_N(t),$$

where the convergence is pointwise and absolute.

Proof. Let $D_{t,n}(\omega) = K(t, \omega) - \phi_n(\omega)\,(K(t, \cdot), \phi_n(\cdot))$. We have

$$\begin{aligned} |f(t) - f_N(t)| &\leq \int_\Omega d\omega\, |g(\omega)||D_{t,n}(\omega)| \\ &\leq \|g\|_2^{1/2} \|D_{t,n}\|_2^{1/2}. \end{aligned}$$

Since $D_{t,n}$ must converge in the mean to zero for all t and $\|g\|_2$ is finite by assumption, the left-hand side converges to zero for all t. □

Remark. When Ω is not of finite measure, many "interesting" kernels K will not be square-integrable as specified in the above. Therefore, we must look to subsets having finite measure or which otherwise have the property that the kernel is square-integrable on them.

Definition. Let $\mathcal{T}$ and Ω be as in the lemma. Let $\mathcal{Q}$ be a subset of Ω such that $K(t,\omega)$ is square-integrable over $\mathcal{Q}$. If $\operatorname{supp}(g)$ is contained in $\mathcal{Q}$ and $f(t) = \int_{\mathcal{Q}} d\mu(\omega)\, K(t,\omega) g(\omega)$, then f is said to be band-limited with band set $\mathcal{Q}$.

Remark. The most common case is $\Omega = \mathcal{T} = \mathbb{R}$, with $\mathcal{Q} = [-W, W]$. In this case we say that f is band-limited with band limit W.

11.5. Theorem. *Let $\mathcal{T}$ and Ω be as in the lemma. Let f be band-limited with band set $\mathcal{Q}$. Let L be a self-adjoint operator on $L^2(\operatorname{supp}(g))$, and suppose $\operatorname{Spec}(L)$ is discrete with all points of finite multiplicity. Let $\{\phi_n\}$ be the normalized eigenfunctions for L, and suppose $\phi_n(\omega) = K(t_n, \omega)$ for some set $\{t_n \in \mathcal{T}\}$. Then, with $f_N(t)$ defined as above, $f(t) = \lim_{N\to\infty} f_N(t)$ for all t.*

Proof. $\{\phi_n\}$ is clearly an orthonormal basis for $L^2(\operatorname{supp}(g))$. By the band-limiting assumption, K is square-integrable over $\mathcal{Q}$. Therefore the lemma applies directly. $\square$

Remark. Now we are in a position to prove the special case sampling theorems discussed above.

11.6. Corollary. *Let f be a band-limited function on $[0,1]$, with band limit M. Let $t_n = n/(2M+1)$. Then the following equality holds pointwise.*

$$f(t) = \frac{1}{2M+1} \sum_{|n|\le M} f\left(t_n \mod 1\right) \frac{\sin((2M+1)\pi(t-t_n))}{\sin(\pi(t-t_n))}.$$

Proof. Let $\mathcal{T}$ be the interval $[0,1]$ and Ω be $\mathbb{Z}$. Let $\mathcal{Q}$ be $\mathbb{Z}_{2M+1}$, identified with the finite set $\{-M, -(M-1), \ldots, M-1, M\}$, which is the band set for band-limited f with band limit M. Let $K(t,m) = \exp(2\pi i m t)$ be the kernel of the Fourier transform on $[0,1]$. Then K is continuous and square-summable on $\mathcal{Q}$ for all t. Let L be the left shift operator on $\mathcal{Q}$, with the identification $M+1 = -M$. Then the spectrum of L is the set $\{t_n = n/(2M+1) : n \in \mathbb{Z}\}$, where each point has multiplicity one. The normalized eigenfunctions are $\phi_n(m) = \exp(2\pi i m t_n)/\sqrt{(}2M+1)$, and $\phi_n(m) = K(t_n, m)$. Therefore the lemma applies, the sum which occurs is actually finite, and we need only compute $S_n(t)$. We have

$$\begin{aligned} S_n(t) &= \sum_{|m|\le M} \exp(2\pi i m(t-t_n)) \\ &= \cos(2M\pi(t-t_n)) + \cot(\pi(t-t_n))\sin(2M\pi(t-t_n)) \\ &= \frac{\sin((2M+1)\pi(t-t_n))}{\sin(\pi(t-t_n))}. \end{aligned}$$

$\square$

11.7. Corollary. *Let $f \in L^2(\mathbb{R})$ be band-limited with band limit W. Then the following holds,*

$$f(t) = \lim_{N->\infty} \sum_{|n|\le N} f\left(\frac{n}{2W}\right) S(t - n/2W),$$

with

$$S(t) = \frac{\sin(2\pi W t)}{2\pi W t},$$

Proof. Let $\mathcal{T}$ and Ω be $\mathbb{R}$. Let $\mathcal{Q}$ be the interval $[-W, W]$. Let $K(t, \omega) = \exp(2\pi i \omega t)$ be the kernel of the Fourier transform on $\mathbb{R}$. Then K is square-integrable over $\mathcal{Q}$. Let $L = -id/d\omega$ be the shift on $[-W, W]$, with the identification $W = -W$. The spectrum of L is the set $\{t_n = n/2W : n \in \mathbb{Z}\}$. The lemma applies, and we compute S_n as follows.

$$\begin{aligned} S_n(t - t_n) &= \frac{1}{2W} \int_{-W}^{W} d\omega \, \exp(2\pi i \omega (t - t_n)) \\ &= \frac{\sin(2\pi W(t - t_n))}{2\pi W(t - t_n)}. \end{aligned}$$

□

Remark. Clearly a host of such sampling theorems present themselves at this point. The following is a nice application to Hankel transforms, both on the finite and the infinite domain.

Definition. Let $f(t)$ be in $L^2([0, 1]; tdt)$. The ν-Hankel transform of f is a function defined on the positive integers, $g_m = \int_0^1 tdt\, f(t) J_\nu(j_{\nu,m} t)$, $m \in \{1, 2, \ldots\}$. As a simple consequence of this definition we have

$$f(t) = \sum_{m=1}^{\infty} g_m \frac{2J_\nu(j_{\nu,m} t)}{J_{\nu+1}(j_{\nu,m})^2}.$$

11.8. Corollary. *Let $f(t)$, g_m be a Hankel transform pair as in the definition. Suppose that g_m is zero for $m > M$. Then $f(t)$ is smooth with $f(1) = 0$ and*

$$f(t) = \sum_{k=1}^{M-1} f(j_{\nu,k}/j_{\nu,M}) S_k(t),$$

with

$$S_k(t) = \frac{2}{j_{\nu,M}^2} \sum_{m=1}^{M-1} \frac{J_\nu(j_{\nu,k} j_{\nu,m}/j_{\nu,M})}{|J_{\nu+1}(j_{\nu,k})|^2 |J_{\nu+1}(j_{\nu,m})|^2} J_\nu(t j_{\nu,m}).$$

Proof. The smoothness and the property $f(1) = 0$ are obvious from the representation of $f(t)$ as a finite sum over Bessel functions. Let $\mathcal{T}$ be the interval, and Ω be $\{1, 2, \ldots\}$. Let $\mathcal{Q}$ be the finite set $\{1, 2, \ldots, M\}$. We must make $\mathcal{Q}$ into a Hilbert space. This we do by choosing a weight function $\mu(m)$ and taking $l^2(\mathcal{Q}; \mu)$. The choice we make is $\mu(m) = \frac{\sqrt{2}}{j_{\nu,M}} |J'_\nu(j_{\nu,m})|^{-2} = \frac{\sqrt{2}}{j_{\nu,M}} |J_{\nu+1}(j_{\nu,m})|^{-2}$. This is precisely the set of Gaussian quadrature weights for the orthonormal set $\phi_n(m) = \frac{\sqrt{2}}{j_{\nu,M}} \frac{J_\nu(j_{\nu,n} j_{\nu,m}/j_{\nu,M})}{|J_{\nu+1}(j_{\nu,n})|}$, $n = 1, 2, \ldots, M-1$. This orthonormal set satisfies the conditions of theorem, in that it is the restriction of the Hankel transform kernel to the finite set of points $\{j_{\nu,1}/j_{\nu,M}, \ldots, j_{\nu,M-1}/j_{\nu,M}\}$, up to a trivial normalization factor. The self-adjoint operator which is diagonal in this basis is the coordinate operator itself, with eigenvalues $j_{\nu,k}/j_{\nu,M}$, $k = 1, \ldots, M-1$. Applying the theorem we get

$$\begin{aligned} S_n(t) &= (J_\nu(t j_{\nu,\cdot}), \phi_n(\cdot)) \\ &= \sum_{m=1}^{M-1} \mu(m) \frac{\sqrt{2}}{j_{\nu,M}} \frac{J_\nu(j_{\nu,n} j_{\nu,m}/j_{\nu,M})}{|J_{\nu+1}(j_{\nu,n})|^2} J_\nu(t j_{\nu,m}) \\ &= \frac{2}{j_{\nu,M}^2} \sum_{m=1}^{M-1} \frac{J_\nu(j_{\nu,n} j_{\nu,m}/j_{\nu,M})}{|J_{\nu+1}(j_{\nu,m})|^2 |J_{\nu+1}(j_{\nu,n})|^2} J_\nu(t j_{\nu,m}). \end{aligned}$$

□

Definition. Let $f(t)$ be in $L^2(\mathbb{R}; tdt)$. The Hankel transform of f is the function $g(\omega)$ given by $g(\omega) = \int_0^\infty tdt\, J_\nu(\omega t) f(t)$. Note that the Hankel transform is its own inverse.

11.9. Corollary. *Let $f(t)$, $g(\omega)$ be a Hankel transform pair. Suppose that* $\operatorname{supp}(g)$ *is contained in* $[0, W]$. *Then*

$$f(t) = \lim_{N\to\infty} \sum_{n=1}^{N} f(j_{\nu,n}/W) S_n(t),$$

with

$$S_n(t) = \frac{2}{j_{\nu,n}^2 - t^2W^2} \frac{J_\nu(tW)}{J_{\nu+1}(j_{\nu,n})}.$$

Proof. Let $\mathcal{T}$ and Ω be $\mathbb{R}$. Let $\mathcal{Q}$ be the interval $[0, W]$. The measure on Ω and thus on $\mathcal{Q}$ is $\omega d\omega$. Let L be the Bessel differential operator with fixed ν on $[0, W]$ with Dirichlet boundary conditions $g(0) = g(W) = 0$. Then the eigenfunctions are $\phi_n(\omega) = J_\nu(t_n\omega)$ with $t_n = j_{\nu,n}/W$, $n = 1, 2, \ldots$. The normalization integral is known,

$$\int_0^W \omega d\omega\, |J_\nu(\omega j_n/W)|^2 = \frac{W^2}{2} |J_{\nu+1}(j_n)|^2.$$

Furthermore we have

$$\int_0^1 xdx\, J_\nu(\alpha x) J_\nu(x j_{\nu,n}) = \frac{J_\nu(\alpha) J_{\nu+1}(j_{\nu,n})}{j_{\nu,n}^2 - \alpha^2}.$$

Therefore

$$\begin{aligned} S_n(t) &= \frac{\int_0^W \omega d\omega\, J_\nu(\omega t) J_\nu(\omega j_{\nu,n}/W)}{\int_0^W \omega d\omega\, |J_\nu(\omega j_{\nu,n}/W)|^2} \\ &= \frac{2}{j_{\nu,n}^2 - t^2W^2} \frac{J_\nu(tW)}{J_{\nu+1}(j_{\nu,n})}. \end{aligned}$$

□

Remark. For a nice discussion of sampling in the general context, see [Kra59].

11.4 Discrete Fourier Transform

Remark. The sampling theorem for the interval $[0, 1]$ can be used to define a discrete version of the Fourier transform for band-limited functions. The idea is that one should be able to express the Fourier transform of a band-limited function in terms of the sampled values of the function. This is an easy computation, given the sampling theorem that we have already proven.

11.10. Theorem. *Let $f(t)$ be a band-limited function on $[0,1]$ with band limit M. Then we have, for $m = -M, \ldots, M$ and $t_n = n/(2M+1) \mod 1$,*

$$\widehat{f}(m) = \frac{1}{2M+1} \sum_{|n| \le M} f(t_n) \exp(-2\pi i m t_n).$$

Proof.

$$\begin{aligned}
\widehat{f}(m) &= \frac{1}{2M+1} \sum_{|n| \le M} f(t_n) \int_0^1 dt \exp(-2\pi i m t) \sum_{|k| \le M} \exp(2\pi i k(t - t_n)) \\
&= \frac{1}{2M+1} \sum_{|n| \le M} \sum_{|k| \le M} f(t_n) \exp(-2\pi i k t_n) \int_0^1 dt \exp(2\pi i (m+k) t) \\
&= \frac{1}{2M+1} \sum_{|n| \le M} f(t_n) \exp(-2\pi i m t_n).
\end{aligned}$$

□

Remark. This discrete representation for the Fourier transform of a band-limited function is called the discrete Fourier transform. Because it can be implemented directly on a computer, it is very often used. Often it is useful even when the functions of interest are not strictly band-limited; however, one must then be aware that the discrete transform is now only an approximation to the Fourier transform. Treating a function which is not band-limited as if it were introduces so-called aliasing errors.

Remark. Perhaps the main reason that the discrete Fourier transform is so important is that there exists an algorithm for computing it with a number of operations of order $M \log M$, as opposed to the naive application of the formula, which would take order M^2 operations. One sophisticated discussion of the fast Fourier transform is given by Auslander and Tolimieri [AT79].

11.11. Theorem.

Proof. □

11.5 Discrete Hankel Transform

11.12. Theorem. *Let $f(t)$ be a function on the unit interval $[0,1]$. Let g_m be the finite ν-Hankel transform of f,*

$$g_m = \int_0^1 tdt\, J_\nu(j_{\nu,m} t) f(t),$$
$$f(t) = \sum_{m=1}^{\infty} \frac{2 J_\nu(j_{\nu,m} x)}{J_{\nu+1}(j_{\nu,m})^2} g_m.$$

Suppose that f is band-limited in the sense that $g_m = 0$, $m > M$. Then we have

$$g_m = \frac{2}{j_{\nu,M}^2} \sum_{k=1}^{M-1} f\left(\frac{j_{\nu,k}}{j_{\nu,M}}\right) \frac{J_\nu(j_{\nu,m} j_{\nu,k} / j_{\nu,M})}{J_{\nu+1}(j_{\nu,k})^2}.$$

Proof. We compute, using the sampling theorem for the finite Hankel transform.

$$
\begin{aligned}
g_m &= \int_0^1 tdt\, J_\nu(tj_{\nu,m})f(t) \\
&= \int_0^1 tdt\, J_\nu(tj_{\nu,m}) \sum_{k=1}^{M-1} f(j_n/j_M)S_k(t) \\
&= \frac{1}{j_{\nu,M}^2} \sum_{m=1}^{M-1} f(j_{\nu,n}/j_{\nu,M}) \frac{J_\nu(j_{\nu,m}j_{\nu,k}/j_{\nu,M})}{J_{\nu+1}(j_{\nu,m})^2}
\end{aligned}
$$

□

Part II

Examples

Chapter 12

$\mathbb{R}$

12.1 Representations

Remark. By the general results on abelian groups, we know that all irreducible representations of $\mathbb{R}$ are one dimensional. The irreducible representations are themselves the characters.

12.1. Lemma. *The only continuous solutions of the functional equation $f(x+y) = f(x)f(y)$ are of the form $f(x) = \exp ax$, $a \in \mathbb{C}$.*

Proof. First note that any solution to the given functional equation must be infinitely differentiable, which we prove as follows. Let $f(x)$ be a continuous solution which is not everywhere zero, and let $\phi(x)$ be any infinitely differentiable function with $\int f(x)\phi(x) = c \neq 0$. We compute $f(y)c = f(y)\int f(x)\phi(x) = \int f(x+y)\phi(x) = \int f(x)\phi(x-y)$; the right-hand side is infinitely differentiable, and therefore so is the left-hand side. Therefore we can assume that any continuous solution is infinitely differentiable. So, using the obvious fact that $f(0) = 1$, we take a limit of the functional equation to obtain $f'(x) = f'(0)f(x)$. But the solutions to this equation are given by the one-parameter family $f(x) = \exp ax$. $\square$

Remark. Given this result, we see that the unitary irreducible representations of $\mathbb{R}$ are given by the characters $\{\chi_\nu(x) = \exp(i\nu x) \,:\, \nu \in \mathbb{R}\}$, and so $\widehat{\mathbb{R}} \cong \mathbb{R}$.

12.2 Fourier Analysis

Remark. We have previously introduced the classical Fourier transform in the context of harmonic analysis on $\mathbb{R}^n$. From the standpoint of representation theory, the Fourier integral is the explicit realization of the direct-integral decomposition

$$L^2(\mathbb{R}) \cong \int^{\oplus} \chi_\nu dE(\nu).$$

Remark. The following application of harmonic analysis seems at first out of place, but it is actually an example of a very general circumstance. See [Fur73].

12.2. Theorem (Central Limit). *Let $\{X_n\}$ be a collection of identical random variables with density $f(x)$. Without loss, suppose the mean is zero and the standard deviation is one for each. Let $\{Y_n\}$ be the collection of random variables defined by $Y_n = (X_1 + \cdots + X_n)/\sqrt{n}$. Then*

$$P(a \le Y_n \le b) \sim (2\pi)^{-1/2} \int_a^b dx\, \exp(-x^2/2), \quad n \to \infty.$$

Proof. Using the assumption about the mean and the standard deviation of each X_n, we have

$$\begin{aligned} \hat{f}(0) &= 1, \\ \hat{f}'(0) &= 0, \\ \hat{f}''(0) &= -4\pi^2. \end{aligned}$$

Therefore

$$\begin{aligned} \hat{f}(s/\sqrt{n}) &\sim 1 - 2\pi^2 s^2/n, \\ \hat{f}(s/\sqrt{n})^n &\sim \exp(-2\pi^2 s^2), \quad n \to \infty. \end{aligned}$$

Recall that the density for Y_n is given by the n-fold convolution of the density for $X_n/\sqrt{n}$, and so this asymptotic result gives the density for Y_n in the limit. Precisely, let d_n be the density for Y_n, then for any Schwarz function $g(x)$ we have

$$\lim_{n\to\infty} \int d_n(x) g(x)\, dx = \int \exp(-2\pi^2 s^2)\hat{g}(s)\, ds = (2\pi)^{-1/2} \int \exp(-x^2/2) g(x)\, dx.$$

Then the result follows by the density of Schwarz functions □

Remark. The following discussion gives an example of the trace formula for a compact domain. The result is classical, but the trace formula approach has a wide generalization.

12.3. Theorem (Poisson Summation). *Let $f(x)$ be a Schwarz function on $\mathbb{R}^n$. Then*

$$\sum_{a\in\mathbb{Z}^n} f(x+a) = \sum_{a\in\mathbb{Z}^n} \hat{f}(a) \exp(2\pi i a \cdot x).$$

Proof. Classical: Calculate the Fourier expansion of the left hand side,

$$\begin{aligned} \hat{f}(a) &= \int f(x) \exp(-2\pi i a \cdot x) dx \\ &= \sum_{b\in\mathbb{Z}^n} \int_{[0,1]^n} f(x+b) \exp\left(-2\pi i a \cdot (x+b)\right) \\ &= \sum_{b\in\mathbb{Z}^n} \int_{[0,1]^n} f(x+b) \exp\left(-2\pi i a \cdot x\right) \\ &= \int_{[0,1]^n} \exp\left(-2\pi i a \cdot x\right) \sum_{b\in\mathbb{Z}^n} f(x+b). \end{aligned}$$

To complete the theorem, we need to know that the Fourier series converges everywhere. This follows from standard results on Fourier series, since the function is smooth and the domain is compact. □

Proof. Trace Formula: Define an integral operator on $L^2\left(\mathbb{R}^n/\mathbb{Z}^n\right)$ by

$$(L_f g)(x) = (f * g)(x), \quad g \in L^2\left(\mathbb{R}^n/\mathbb{Z}^n\right).$$

This is a Hilbert-Schmidt operator defined on a compact domain. Therefore it has a smooth kernel, given by

$$K_f(x,y) = \sum_{b\in\mathbb{Z}^n} f(x-y-b),$$

and its trace is given by

$$\mathrm{Tr}\left(L_f\right) = \int_{\mathbb{R}^n/\mathbb{Z}^n} K_f(x,x)dx = \sum_{a\in\mathbb{Z}^n} f(a).$$

On the other hand, the eigenvalues of L_f are precisely $\left\{\hat{f}(b), b \in \mathbb{Z}^n\right\}$, with eigenvectors $\{\exp(2\pi i b\cdot x)\}$. Therefore

$$\mathrm{Tr}\left(L_f\right) = \sum_{b\in\mathbb{Z}^n} \hat{f}(b).$$

Equating the two expressions gives the result. □

Remark. The assumption that $f(x)$ is a Schwarz function can presumably be weakened. However, this is apparently not trivial. It is worth noting that there exist examples of $f \in L^1\left(\mathbb{R}\right)$ for which the Poisson summation formula does not hold. See [Kat76, p. 130].

Chapter 13

$\mathbb{R}^*$

Remark. Let $\mathbb{R}^*$ be the multiplicative group of positive real numbers. This is an abelian Lie group. Recall that a Haar measure on $\mathbb{R}^*$ is given by dx/x. The characters are $\chi_t(x) = x^{it}$, $t \in \mathbb{R}$. These results can be obtained most easily by mapping $\mathbb{R}$ to $\mathbb{R}^*$ using the logarithm and using the results for $\mathbb{R}$.

Definition. Let $f(x)$ be a complex-valued function on $\mathbb{R}^*$. Define the Mellin transform of f at $s \in \mathbb{C}$ to be

$$(Mf)(s) = \int f(x)x^s \frac{dx}{x}.$$

Remark. In the case of the real line, $\mathbb{R}$, the set of characters $\{\exp(ikx)\}$ gives a spectral resolution of the invariant differential operator $(d/dx)^2$. Similarly for $\mathbb{R}^*$, the set of characters $\{x^{it}\}$, $t \in \mathbb{R}$, gives a spectral resolution of the invariant operator $(x\, d/dx)^2$.

Remark. The Mellin transform for real values of s is related to the Mellin transform for imaginary values of s in the same way that the Laplace transform is related to the Fourier transform.

Chapter 14

$\mathrm{Affine}\,(\mathbb{R}) = \mathbb{R}^* \ltimes \mathbb{R}$

Definition. The group of affine transformations of the real line is given by maps $g(a,b) : x \mapsto ax + b$; $a, b \in \mathbb{R}, a > 0$. This is a nonabelian Lie group. It is an example of a semi-direct product because $\mathbb{R}^*$ has a nontrivial action on $\mathbb{R}$, which is seen in the multiplication law $g(a_1, b_1)g(a_2, b_2) = g(a_1 a_2, a_1 b_2 + b_1)$.

Remark. We have previously displayed the Haar measure(s) on $\mathrm{Affine}\,(\mathbb{R})$. The measure $d\mu_R = da\,db/a$ is right-invariant, and the measure $d\mu_L = da\,db/a^2$ is left-invariant. $\mathrm{Affine}\,(\mathbb{R})$ is not unimodular.

Remark. The multiplicative subgroup $\mathbb{R}^*$ of $\mathrm{Affine}\,(\mathbb{R})$ has the representations we have already seen, with characters $\xi_t(g(a,0)) = a^{it}$. For $\nu \in \mathbb{R}$ these are unitary irreducible representations.

14.1. Lemma. *Let $\mathcal{H} = L^2\,(\mathbb{R}^*, dx/x)$. Define the following bounded linear maps on $C_0\,(\mathbb{R}^*)$ and extend to them to $\mathcal{H}$,*

$$\rho_+(g(a,b))f(x) = \exp(-ibx)f(ax),$$
$$\rho_-(g(a,b))f(x) = \exp(ibx)f(ax).$$

Then ρ_+ and ρ_- are unitary representations of $\mathrm{Affine}\,(\mathbb{R})$ on $\mathcal{H}$.

Proof. The invariance of the inner-product is a simple computation. Continuity is obvious. □

14.2. Theorem (Gelfand-Naimark). *The representations ρ_+ and ρ_- are irreducible, and the representations $\xi_t(g(a,0)) = a^{it}$, ρ_+, and ρ_- together exhaust the unitary irreducible representations of $\mathrm{Affine}\,(\mathbb{R})$.*

Proof. □

Chapter 15

SO (3)

15.1 Spherical Harmonics

Definition. Define $Y_{lm}(\theta, \phi) = e^{im\phi} P_l^m(\cos\theta)$. These functions occur as eigenvectors of the Laplacian on the sphere S^2, with eigenvalue $-l(l+1)$, which can be demonstrated by elementary separation of variables in spherical coordinates.

15.1. Theorem. *The finite-dimensional space spanned by the complex-valued functions* $\{Y_{lm}(\omega),\ |m| \leq l\}$ *admits a linear representation of the group* SO (3)*,*

$$Y_{lm}(g\omega) = \sum_{|k|\leq l} (A_l(g))_{m,k}\, Y_{lk}(\omega).$$

With the obvious inner product this representation is unitary.

Proof. This is a straightforward exercise. □

Remark. It is well-known that the spherical harmonics exhaust the irreducible unitary representations of SO (3). This is proved for instance in Ref. [Vil68]. However, this is not a general result. One can define representations of SO (n) on higher-dimensional harmonics with a formula like that of the above theorem. It turns out that the set of harmonics defined this way do not exhaust the irreducible representations of SO (n) if $n > 3$. This is also proved in Ref. [Vil68]. In fact, for $n > 3$ the spaces of harmonics so defined exhaust only the set of representations which have a vector which is fixed under an SO $(n-1)$ subgroup.

15.2. Theorem (Addition Formula). *Let* $\omega_1, \omega_2 \in S^2$*. Then*

$$\sum_{m\leq l} Y_{lm}(\omega_1) Y_{lm}(\omega_2)^* = \frac{2l+1}{4\pi} P_l(\omega_1 \cdot \omega_2).$$

Proof. Each side of the equation is invariant upon replacing simultaneously $\omega_1 \mapsto g\omega_1$ and $\omega_2 \mapsto g\omega_2$, where $g \in$ SO (3) acts on the points as usual. (The left-hand side is invariant because the sum on m makes it a scalar in the representation space.) So the two sides must be proportional since the scalar representation of the group is one-dimensional. To determine the constant of proportionality, set $\omega_1 = \omega_2$ and integrate both sides over S^2. □

15.3. Theorem (Funk-Hecke). *Let* $f : [-1,1] \longrightarrow \mathbb{C}$ *be a continuous function. Then*

$$\int_{S^2} f(\omega' \cdot \omega) Y_{lm}(\omega')\, d\omega' = 2\pi Y_{lm}(\omega) \int_{-1}^{1} f(t) P_l(t)\, dt$$

Proof. Using the addition formula it is easy to show that

$$\int_{S^2} Y_{lm}(\omega_1) P_k(\omega_1 \cdot \omega_2)\, d\omega_1 = \frac{4\pi}{2k+1} Y_{lm}(\omega_2)\delta_{lk}.$$

Since f is continuous on the closed interval, it can be uniformly approximated by Legendre polynomials $P_l(t)$. Expand f in this way, and the result follows immediately. $\square$

Chapter 16

$\mathrm{SU}(2)$

16.1 Holomorphic Representations

Let E_{N+1} denote the set of homogeneous polynomials of degree N in the variables z_1, z_2. As we note elsewhere, this space admits a reprsentation of $\mathrm{SL}_2(\mathbb{C})$. Therefore it also admits a representation of $\mathrm{SU}(2)$. By the general theory, since $\mathrm{SU}(2)$ is compact, we know that this representation can be unitarized.

16.2 Characters and Dimensions of Representations

We can use the Weyl character formula and the Weyl dimension formula to obtain information about the representations of $\mathrm{SU}(2)$. In fact, since these are Lie algebraic results, they apply to any group of type A_1 in the Dynkin classification.

Recall that dominant integral weights correspond to the finite-dimensional irreducible $\mathfrak{g}$-modules. These are the representation spaces in which we are interested. For A_1, there is one fundamental weight, say w_1, and the dominant integral weights are $\{mw_1\}$, $m \in \mathbb{Z}$, $m \geq 0$. Consider the $\mathfrak{g}$-module $L(mw_1 + \rho_+)$. The characters of the maximal torus $\mathrm{U}(1)$ are exponentials, the Weyl group is generated by one reflection, and the character of the associated representation is thus given by

$$\chi_m = \frac{\xi_{w_1}^{m+1} + (-1)\xi_{w_1}^{-m-1}}{\xi_{w_1} + (-1)\xi_{w_1}^{-1}}.$$

The dimension of the representation $L(mw_1 + \rho_+)$ is

$$\dim L(mw_1 + \rho_+) = m + 1.$$

Chapter 17

SU (3)

17.1 Dimensions of Representations

Pick a dominant integral weight $\lambda = m_1 w_1 + m_2 w_2$. The positive roots of A_2 can be written $R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$, and α_1, α_2 have the same length. Using the explicit form of the Weyl dimension formula we have

$$\dim L(\lambda + \rho_+) = \frac{(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)}{2}.$$

The dimensions are thus $\{1, 3, 6, 8, 10, 15, 24, 27, \ldots\}$.

Chapter 18

$\mathrm{SL}_2(\mathbb{C})$

18.1 Introduction

Remark. It is worth pointing out that $\mathrm{SL}_2(\mathbb{C})$ is the double cover of the proper orthochronous Lorentz group.

18.2 Finite-Dimensional Representations

Recall that a function $p(z)$ is called homogeneous of degree n if $p(\lambda z) = \lambda^n p(z)$ for all λ. Let E_{N+1} denote the set of homogeneous polynomials of degree N in the variables z_1, z_2. Note that $\dim E_{N+1} = N+1$, explaining the notation. For such a polynomial, we can write

$$p(z_1, z_2) = \sum_{k=0}^{N} c_k z_1^k z_2^{N-k}.$$

Let s be an element of $\mathrm{SL}_2(\mathbb{C})$

$$s = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which we take to act on the two complex-dimensional space of the variables z_1, z_2. This defines an action on the space E_{N+1} by composition,

$$(\rho_N(s)p)(z_1, z_2) = p(s^{-1}(z_1, z_2) = p(dz_1 - cz_2, -bz_1 + az_2).$$

Clearly $\rho_N(st) = \rho_N(s)\rho_N(t)$ for $s, t \in \mathrm{SL}_2(\mathbb{C})$.

18.1. Theorem. *The set of irreducible finite-dimensional representations of* $\mathrm{SL}_2(\mathbb{C})$ *is exhausted by spaces of polynomials as given above.*

18.3 Gelfand Method

Remark. A systematic construction of the representations for $\mathrm{SL}_2(\mathbb{C})$, and other groups, was obtained by Gelfand and collaborators [GGV66]. The following discussion gives a few of the basic ingredients in this construction.

Definition. Suppose κ_1, κ_2 are two complex numbers such that $\kappa_1 - \kappa_2 \in \mathbb{Z}$. Write $\chi = (\kappa_1, \kappa_2)$. For each such χ define the vector space D_χ to be the space of functions $\phi : \mathbb{C}/\{0\} \longrightarrow \mathbb{C}$ with the properties

- $\phi(z, \overline{z}) \in C^\infty(\mathbb{C}/\{0\})$,
- $z^{\kappa_1 - 1}\overline{z}^{\kappa_2 - 1}\phi(-z^{-1}, -\overline{z}^{-1}) \in C^\infty(\mathbb{C}/\{0\})$.

Topologize D_χ by uniform convergence, together with all derivatives, on compact subsets of $\mathbb{C}/\{0\}$, making D_χ a locally convex topological vector space.

Definition. Define an action of $\mathrm{SL}_2(\mathbb{C})$ on D_χ by

$$\rho_\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi(z) = (-bz + d)^{\kappa_1 - 1}(-\overline{b}\overline{z} + \overline{d})^{\kappa_2 - 1}\phi\left(\frac{az - c}{-bz + d}\right).$$

18.2. Lemma. *The above defines a representation of* $\mathrm{SL}_2(\mathbb{C})$ *on* D_χ *as a locally convex topological vector space, i.e. is continuous with the given topologies.*

Remark. It is sometimes useful to consider another realization of D_χ as a space of functions of the sphere $|\omega_1|^2 + |\omega_2|^2 = 1$, $(\omega_1, \omega_2) \in \mathbb{C}^2$, given by the identification $f(\omega_1, \omega_2) = \omega_1^{\kappa_1 - 1}\omega_2^{\kappa_2 - 1}\phi(\omega_1/\omega_2)$.

Remark. When κ_1 and κ_2 are both non-negative integers, D_χ has an invariant subspace, the subspace of polynomials $p(z, \overline{z})$ of order at most κ_1 in z and κ_2 in $\overline{z}$. This is easy to see from the definition of the action of the representation. This special case is related to the unitary complementary series, whereas the generic case gives the unitary principal series. The following gallery of invariant bilinear functionals illustrates these different cases.

18.3. Theorem. *Let* D_χ *and* D_ζ *denote representation spaces as above, writing* $\chi = (\kappa_1, \kappa_2)$. *Then an invariant bilinear functional* $B : D_\chi \times D_\zeta \longrightarrow \mathbb{C}$ *exists in the following cases.*

1. $\zeta = \chi$, κ_1 *and* κ_2 *not non-negative integers:* $B(\phi, \psi) = -\frac{1}{4}\int dz_1 d\overline{z}_1 dz_2 d\overline{z}_2 (z_1 - z_2)^{-\kappa_1 - 1}(\overline{z}_1 - \overline{z}_2)^{-\kappa_2 - 1}\phi\psi$,
2. $\zeta = \chi$, $\kappa_1, \kappa_2 = 0, 1, 2, \ldots$*:* $B(\phi, \psi) = \frac{i}{2}\int dz d\overline{z}\left(\partial^{\kappa_1}\overline{\partial}^{\kappa_2}\phi(z, \overline{z})\right)\psi(z, \overline{z})$,
3. $\zeta = -\chi$*:* $B(\phi, \psi) = \frac{i}{2}\int dz d\overline{z}\phi(z, \overline{z})\psi(z, \overline{z})$,
4. $\zeta = (\kappa_1, -\kappa_2)$, $\kappa_1 = 1, 2, \ldots$*:* $B(\phi, \psi) = \frac{i}{2}\int dz d\overline{z}\left(\partial^{\kappa_1}\phi(z, \overline{z})\right)\psi(z, \overline{z})$,
5. $\zeta = (-\kappa_1, \kappa_2)$, $\kappa_2 = 1, 2, \ldots$*:* $B(\phi, \psi) = \frac{i}{2}\int dz d\overline{z}\left(\overline{\partial}^{\kappa_2}\phi(z, \overline{z})\right)\psi(z, \overline{z})$.

Remark. The first two cases are related by the following generalized function identity:

$$\lim_{\kappa_1, \kappa_2 \to k_1, k_2 \in \mathbb{Z}} \frac{(z_1 - z)^{-\kappa_1 - 1}(\overline{z}_1 - \overline{z})^{-\kappa_2 - 1}}{\Gamma(-\kappa_1/2 - \kappa_2/2 + |\kappa_1 - \kappa_2|/2)} = \delta^{(k_1, k_2)(z_1 - z)}.$$

Remark. Invariant Hermitian functionals on D_χ can be constructed from invariant bilinear functionals in the obvious manner, $h(\phi,\psi) = B(\phi,\overline{\psi})$, when B is invariant under $\rho_\chi \otimes \rho_{\overline{\chi}}$.

18.4. Theorem. *(ρ_χ, D_χ) admits a positive definite invariant Hermitian functional if and only if one of the following holds:*

- $\kappa_1 = -\overline{\kappa_2} = \frac{1}{2}(k+iv), k \in \mathbb{Z}$: $h(\phi,\psi) = \frac{i}{2}\int dz d\overline{z}\, \phi(z_1,\overline{z}_1)\psi(z_2,\overline{z}_2)$,
- $\kappa_1 = \overline{\kappa_2} = r \in (-1,1), r \neq 0$: $h(\phi,\psi) = -\frac{1}{4\Gamma(-r)}\int dz_1 d\overline{z}_1 dz_2 d\overline{z}_2 |z_1 - z_2|^{-2r-2}\phi(z_1,\overline{z}_1)\psi(z_2,\overline{z}_2)$.

Remark. Completing the first case to a Hilbert space gives the unitary principal series. Completing the second case to a Hilbert space gives the unitary complementary series.

18.4 Unitary Principal Series

Remark. The above construction leads to the principal and complementary series. Because of the importance of the unitary principal series, we explicitly state the existence and uniqueness results. The following form of the $\mathrm{SL}_2(\mathbb{C})$ action is equivalent to that given above, in the principal series case.

Definition. For $f(z) \in L^2(\mathbb{C})$, define the following action on f,

$$\mathcal{P}^{k,iv}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = |-bz+d|^{-2-iv}\left(\frac{-bz+d}{|-bz+d|}\right)^{-k} f\left(\frac{az-c}{-bz+d}\right).$$

This defines a unitary representation of $\mathrm{SL}_2(\mathbb{C})$ on $L^2(\mathbb{C})$, for any $k \in \mathbb{Z}$ and $v \in \mathbb{R}$.

18.5. Theorem. *$\mathcal{P}^{k,iv}$ is an irreducible unitary representation of $\mathrm{SL}_2(\mathbb{C})$. Moreover, $\mathcal{P}^{k,iv}$ is unitarily equivalent to $\mathcal{P}^{-k,-iv}$.*

Proof. See [Kna86, p. 33]. □

18.6. Theorem. *The above representations exhaust the unitary principal series for $\mathrm{SL}_2(\mathbb{C})$.*

18.5 Plancherel Inversion Formula

18.7. Theorem. *Let f be a smooth function of compact support on $\mathrm{SL}_2(\mathbb{C})$. Then we have the Plancherel inversion formula,*

$$f(1) = \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} dv\, \mathrm{Tr}\left(\mathcal{P}^{k,iv}(f)\right)\left(k^2+v^2\right),$$

which implicitly defines the Plancherel measure on the space $\mathbb{Z}\times\mathbb{R}$.

Chapter 19

$\mathrm{SL}_2(\mathbb{R})$

19.1 Introduction

19.1. Theorem. *Every finite-dimensional unitary representation of* $\mathrm{SL}_2(\mathbb{R})$ *is trivial.*

19.2 Gelfand Method

Remark. Again we consider the method of construction discussed by Gelfand and collaborators [GGV66].

Definition. Let $s \in \mathbb{C}$ and $\epsilon = 0, 1$, and write $\chi = (s, \epsilon)$. For each such χ define the vector space D_χ of functions $\phi : \mathbb{R}/\{0\} \longrightarrow \mathbb{C}$ with the properties

- $\phi(x) \in C^\infty(\mathbb{R}/\{0\})$,
- $|x|^{s-1}\mathrm{sgn}_\epsilon(x)\phi(-1/x) \in C^\infty(\mathbb{R}/\{0\})$.

Topologize D_χ by uniform convergence, together with all derivatives, on compact subsets of $\mathbb{R}/\{0\}$, making D_χ a locally convex topological vector space.

19.2. Lemma. *Let* $\phi \in D_\chi$. *Then the behaviour of* $\phi(x)$ *for* $|x| \to \infty$ *is* $\phi(x) = \mathcal{O}(|x|^{s-1})$.

Definition. Define an action of $\mathrm{SL}_2(\mathbb{R})$ on D_χ by

$$\rho_\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi(x) = |-bx+d|^{s-1}\mathrm{sgn}_\epsilon(-bx+d)\phi\left(\frac{ax-c}{-bx+d}\right).$$

19.3. Lemma. *The above defines a representation of* $\mathrm{SL}_2(\mathbb{R})$ *on* D_χ *as a locally convex topological vector space, i.e. is continuous with the given topologies.*

Remark. If χ is such that $\rho_\chi\phi = (-bx+d)^{s-1}\phi\left(\frac{ax-c}{-bx+d}\right)$, then the representation ρ_χ is called analytic.

Remark. It is sometimes useful to consider another realization of D_χ as a space of functions on the circle, given by the identification $f(\theta) = |\sin\theta|^{s-1}\mathrm{sgn}_\epsilon(\sin\theta)\phi(\cot\theta)$.

19.4. Theorem. *(ρ_χ, D_χ) admits a positive definite invariant Hermitian functional if and only if one of the following holds:*

- $s = -\overline{s} = ir$, $r \in \mathbb{R}$, $r \neq 0$: $h(\phi, \psi) = \int \phi(x)\overline{\psi}(x)dx$,

- $|s| < 1$, $s \neq 0$, $\epsilon = 0$: $h(\phi, \psi) = \sigma \int |x_1 - x_2|^{-s-1}\phi(x_1)\overline{\psi}(x_2)dx_1dx_2$, *where* $\sigma = -1$ *when* $s > 0$ *and* $\sigma = 1$ *otherwise.*

Remark. The above theorem shows the existence of the principal and complementary unitary series. However, in the case of $\mathrm{SL}_2(\mathbb{R})$ further unitary representations exist. These are based on the existence of proper invariant subspaces of D_χ for certain values of χ.

Definition. Let $s = 0, 1, \ldots$. Define the operators A_+, A_- as follows, when $\phi(x)$ is sufficiently smooth.

$$A_+\phi(x) = \phi_+^{(s)}(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\phi^{(s)}(x')dx'}{x' - x - i0},$$
$$A_-\phi(x) = \phi_-^{(s)}(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\phi^{(s)}(x')dx'}{x' - x + i0}.$$

19.5. Lemma. *Let $s = 0, 1, \ldots$ and A_+, A_- be as above. Let $(\cdot, \cdot)$ be the standard L^2 inner-product. Then the following define independent invariant Hermitian functionals (possibly degenerate) on D_χ.*

$$(\phi, \psi)_+ = i^{-s}\left(A_+\phi, \psi\right),$$
$$(\phi, \psi)_+ = i^{s}\left(A_-\phi, \psi\right).$$

19.6. Lemma. *Let ρ_χ, D_χ be an analytic representation, so $s = 0, 1, \ldots$ and ϵ is adjusted accordingly. Let $D_s^- \subset D_\chi$ be the nullspace for $(\cdot,\cdot)_+$ and $D_s^+ \subset D_\chi$ be the nullspace for $(\cdot,\cdot)_-$. Let $E_s \subset D_\chi$ be the subspace of polynomials of order at most $s - 1$. Then*

- *D_s^+ and D_s^- are proper invariant subspaces of D_χ,*

- *$D_s^+ \cap D_s^- = E_s$,*

- *$D_\chi/E_s = D_s^+/E_s \oplus D_s^-/E_s$,*

- *The invariant hermitian functionals given above are nondegenerate and positive-definite on the respective spaces, so that $F^+ \equiv A_+(D_s^+/E_s) \cong D_s^+/E_s$ and $F^- \equiv A_-(D_s^-/E_s) \cong D_s^-/E_s$.*

Remark. The spaces F^+ and F^- have equivalent representations as spaces of functions holomorphic in the upper half-plane and lower half-plane repsectively.

19.3 Unitary Principal Series

19.7. Theorem. *For $f(x) \in L^2(\mathbb{R})$, define the following actions on f,*

$$\mathcal{P}^{\pm,iv}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = \sigma(\pm)|-bx+d|^{-1-iv} f\left(\frac{ax-c}{-bx+d}\right),$$

$$\sigma(s) = \begin{cases} 1 & s = +, \\ \operatorname{sgn}(-bx+d) & s = -. \end{cases}$$

These define unitary representations of $\mathrm{SL}_2(\mathbb{R})$ *on* $L^2(\mathbb{R})$*, for any* $v \in \mathbb{R}$.

19.8. Theorem. *The above representations exhaust the unitary principal series for* $\mathrm{SL}_2(\mathbb{R})$.

19.4 Unitary Discrete Series

Definition. Let D_n^+, $n \geq 2$, be the Hilbert space of holomorphic functions on the upper half-plane, with the norm

$$\|f\|^2 = \iint_{y>0} |f(z)|^2 y^{n-2} dx dy.$$

19.9. Theorem. *For $f(z) \in D_n^+$, define the following actions on f,*

$$\mathcal{D}^{\pm,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = \sigma(\pm) \circ (-bx+d)^{-n} f\left(\frac{ax-c}{-bx+d}\right),$$

where $\sigma(+)$ is the identity map and $\sigma(-)$ is complex conjugation. These define unitary representations of $\mathrm{SL}_2(\mathbb{R})$ *on D_n^+, for any $n \geq 2$.*

Proof. See [Kna86, p. 35]. □

19.10. Theorem. *The above representations exhaust the unitary discrete series for* $\mathrm{SL}_2(\mathbb{R})$.

19.5 Plancherel Inversion Formula

19.11. Theorem. *Let f be a smooth function of compact support on* $\mathrm{SL}_2(\mathbb{R})$*. Then we have the Plancherel inversion formula,*

$$\begin{aligned} f(1) = & \int_{-\infty}^{\infty} dv \operatorname{Tr}\left(\mathcal{P}^{+,iv}(f)\right) v \tanh(\pi v/2) \\ & + \int_{-\infty}^{\infty} dv \operatorname{Tr}\left(\mathcal{P}^{-,iv}(f)\right) v \coth(\pi v/2) \\ & + \sum_{k=2}^{\infty} 4(k-1) \operatorname{Tr}\left(\mathcal{D}^{+,k}(f) + \mathcal{D}^{-,k}(f)\right). \end{aligned}$$

which implicitly defines the Plancherel measure on the space $\mathbb{Z} \times \mathbb{R} \cup \mathbb{Z}^+$.

19.6 Curious Topology for Lattice Spaces

Consider lattices in $\mathbb{R}^2$; recall that such lattices are by definition subgroups of $\mathbb{R}^2$ isomorphic to $\mathbb{Z}\times\mathbb{Z}$ generated by two linearly independent basis vectors. First note that $\mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_2(\mathbb{Z})$ is equivalent to the space of lattices in $\mathbb{R}^2$, and similarly $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ is equivalent to the space of lattices in $\mathbb{R}^2$ satisfying the condition that each unit cell has unit area. The topology of these spaces is known.

19.12. Theorem. *The spaces of lattices in the plane have the following topologies.*

1. $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ *is homeomorphic to the complement of a trefoil knot in* $\mathbb{R}^3$.
2. $\mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_2(\mathbb{Z})$ *is homeomorphic to the space of unordered triples of points in* $\mathbb{R}^2$ *with fixed center of mass.*

Proof. See [Mil71, p. 84]. $\square$

Chapter 20

Heisenberg Group

Definition. Let N be the group of 3×3 upper-triangular matrices with real entries and with diagonal entries all equal to 1,

$$N = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;:\; a, b, c \in \mathbb{R} \right\}.$$

Let Z be the normal subgroup of N consisting of matrices of the form

$$Z = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;:\; n \in \mathbb{Z} \right\}.$$

Define the Heisenberg group to be the group N/Z.

20.1. Theorem. *The Heisenberg group H is not a matrix group.*

Proof. Follows [Seg95]. Let T be the circle subgroup of H given by matrices of the form

$$T = \left\{ g_t = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;:\; t \in \mathbb{R} \right\}.$$

Suppose that we have a representation ρ, V of H on some finite-dimensional vector space V. Decompose this representation under the action of T, into irreducible subrepresentations ρ_n, V_n. Each V_n is an invariant subspace under the action of H, since T is in the center of H. Explicitly, g_t acts on V_n by multiplication by a phase $e^{-2\pi i n t}$. But each g_t can be written in the form $g_t = uvu^{-1}v^{-1}$ for some $u, v \in H$; therefore $\rho(g_t)$ must act as multiplication by 1, since it has determinant 1. This means that only $n = 0$ can appear in the decomposition of V. Therefore T acts trivially on V. But then ρ cannot be injective. Therefore no finite-dimensional ρ, V can provide a faithful representation. $\square$

Remark. Roughly speaking, there is a circle group inside H which is invisible from the standpoint of finite-dimensional representations.

Remark. H is a central extension of $\mathbb{R}^2$ by the circle group,

$$0 \to T \to H \to \mathbb{R}^2 \to 0.$$

Write the elements of H in the form $u\exp(\xi)$ with $u \in T$ and $\xi \in \mathbb{R}^2$. Let $S(\xi,\eta)$ be the skew form $\xi_1\eta_2 - \xi_2\eta_1$ on $\mathbb{R}^2$. Then the group law is

$$u\exp(\xi)\, v\exp(\eta) = uv\exp(iS(\xi,\eta))\exp(\xi+\eta).$$

20.2. Theorem. *The unique faithful representation of H is the standard one of quantum mechanics, on the space $L^2(\mathbb{R})$, generated by the one-parameter groups,*

$$e^{iaP},\ e^{ibQ},\ e^{2\pi it},$$

where

$$Q = x\cdot,\ P = -i\frac{d}{dx}.$$

Proof. This is the Stone von-Neumann theorem. It is not difficult to prove. See my notes on C^* algebras. ☐

Remark. Pick a complex structure on $\mathbb{R}^2$ compatible with the skew form S defined above. Thus identify $\mathbb{R}^2$ with $\mathbb{C}$. If we define

$$a = \frac{1}{\sqrt{2}}(P+iQ),\ a^\dagger = \frac{1}{\sqrt{2}}(P-iQ),$$

then, on a common domain for P and Q we have $\left[a^\dagger, a\right] = 1$. It can be seen easily that $a^\dagger$ annihilates the unique vector $\Omega = \exp(-\frac{1}{2}x^2)$ and that the set $\{a^n\Omega, n \geq 0\}$ provides an orthonormal basis for $\mathcal{H} = L^2(\mathbb{R})$. Considering scalar multiples of these basis elements, we see that $\mathcal{H}$ has a dense subspace isomorphic to the symmetric algebra $\operatorname{Sym}(\mathbb{C})$. Thinking of $\mathcal{H}$ as the completion of $\operatorname{Sym}(\mathbb{C})$, we call it the oscillator representation.

Remark. The three self-adjoint operators

$$\left\{\frac{1}{2}P^2, \frac{i}{2}(PQ+QP), \frac{1}{2}Q^2\right\},$$

share the common dense domain identified with $\operatorname{Sym}(\mathbb{C})$ above. On this domain they satisfy the commutation relations of $\mathfrak{sl}_2(\mathbb{R})$. The one-parameter unitary groups associated to these operators generate a group which is a double cover of $\mathrm{SL}_2(\mathbb{R})$, which is called the metaplectic group $\mathrm{Mpl}_2(\mathbb{R})$.

Remark. As pointed out by Segal [Seg95, p. 102], this seems related to the following situation. Consider the space of spherically symmetric functions on $\mathbb{R}^n$, and consider the operators on such functions,

$$\{e,h,f\} = \left\{\frac{1}{2}\triangle, r\frac{\partial}{\partial r} + n/2, \frac{1}{2}r^2\right\}.$$

These satisfy the relations of $\mathfrak{sl}_2(\mathbb{R})$,

$$[h,e] = -2e, \quad [h,f] = 2f, \quad [e,f] = h.$$

But these representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ on spaces of spherically symmetric functions do not correspond to any representation of $\mathrm{SL}_2(\mathbb{R})$, and they have no apparent geometric interpretation.

Remark. Define the operator $A = a^\dagger a + 1/2 = \frac{1}{2}(P^2 + Q^2)$. This self-adjoint operator generates the circle group in $\mathrm{SL}_2(\mathbb{R})$. In quantum mechanics $a^\dagger a$ is called the number operator, and A is the Hamiltonian of the harmonic oscillator. See [GS93, p. 75].

Remark. It is worth thinking about the physical connection here, because the appearance of the harmonic oscillator needs to be understood. We began with kinematic information, in the form of the quantum algebra of observables generated from P and Q. The associated group contains a circle group which is invisible to finite-dimensional representations, but which controls the phase information in the (necessarily infinite-dimensional) quantum Hilbert space. All this information is purely kinematical, in the sense that it applies equally well to any quantum mechanical system. Suppose now that we *define* the time-evolution of the system to be the automorphism generated by the action of this circle group; then we have the system which we call the harmonic oscillator. The harmonic oscillator is special precisely because it is the system with dynamics defined in terms of this kinematic information, and this is precisely why it is exactly soluble; the harmonic oscillator is purely representation theoretic.

Remark. One might ask about other exactly soluble quantum mechanical systems, specifically the hydrogen atom. Can the exact solution of the hydrogen atom be understood group-theoretically? It turns out that it can, and indeed $\mathrm{SL}_2(\mathbb{R})$ appears again.

Definition. Define the Schroedinger representation of the Heisenberg group by

$$\begin{aligned}\rho(p,q,t) &= \exp(2\pi i t)\exp\left(2\pi i(qP + pQ)\right), \\ &\equiv \rho(p,q)\exp(2\pi i t),\end{aligned}$$

acting on functions in $L^2(\mathbb{R})$.

Definition. Define the Wigner transform as the map $V : L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}^2)$

$$V(f,g)(p,q) = \int dy\ \exp(2\pi i p y) f\left(y + \frac{1}{2}q\right)\overline{g\left(y - \frac{1}{2}q\right)}.$$

Definition. Let $\phi_0 = \sqrt{2}\exp(-\pi x^2)$. Define the Bargmann transform to be the linear map $B : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{C} - \{0\}; \exp(-\pi|z|^2) dz d\bar{z})$,

$$\begin{aligned}(Bf)(z) &= \exp\left(\frac{\pi}{2}|z|^2\right) V(f,\phi_0) \\ &= \int dx\, f(x)\sqrt{2}\exp\left(2\pi x z - \pi x^2 - \frac{\pi}{2}|z|^2\right).\end{aligned}$$

The integral kernel is called the Bargmann kernel. The space $\mathcal{F} \subset L^2\left(\mathbb{C} - 0; e^{-\pi|z|^2} dz d\bar{z}\right)$ given by

$$\mathcal{F} = \left\{ f(z) \,:\, f(z) \text{ entire}, \int f(z) e^{-\pi|z|^2} dz d\bar{z} < \infty \right\},$$

is called the Fock space. The composition of the Bargmann map and the Schroedinger representation gives a representation of the Heisenberg group on $\mathcal{F}$ called the Bargmann representation; the Bargmann map is a unitary equivalence. See [Fol87].

20.3. Theorem (Groenewald). *Let P_k be the space of real polynomials of degree less than or equal to k on $\mathbb{R}^{2n}$. Then there does not exist a linear map $R : P_4 \longrightarrow \mathcal{B}(\mathcal{S}(\mathbb{R}^n))$ with the properties*

1. $R(\xi_j) = D_j$,
2. $R(x_j) = X_j$,
3. $R(\{A, B\}) = 2\pi i\,[R(A), R(B)]$ *for all* $A, B \in P_3$.

Proof. This is a simple calculation. For such a map we have $R(\xi^2 x^2) = \frac{1}{9}R(\{\xi^3, x^3\}) = \frac{2\pi i}{9}[D^3, X^3]$. Also $R(\xi^2 x^2) = \frac{1}{3}R(\{\xi^2 x, \xi^2 x\}) = \frac{\pi i}{6}[D^2 X + XD^2, X^2 D + DX^2]$. But these are contradictory, as can be seen by applying each operator to the function $f(x) = 1$. □

Remark. This theorem shows that quantization is not a simple functorial operation on the Poisson structure of classical mechanics. The algebra of local flows, generated by such polynomial functions as appear in the theorem, cannot simply be transferred to an operator algebra in the quantum theory. However, it could be argued that this is too much to ask. Perhaps one should not consider all the local one-parameter flows, but only those which are actually global flows. In fact this case is also impossible.

20.4. Theorem (van Hove). *The conclusions of the Groenewald theorem remain true as well if polynomials are replaced everywhere by smooth functions generating global one-parameter flows.*

Proof. See [Got80] and [Fol87, p. 197] for discussion. □

Chapter 21

Integrating Class Functions

Segal gives an exposition of the nice result for calculating the integral of a class function over the group $\mathrm{U}(n)$ [Seg95, p. 86]. Recall that a class function is a function f on a group G satisfying $f(hgh^{-1}) = f(g)$ for all $h, g \in G$.

21.1. Theorem. *Let* $f : \mathrm{U}(n) \longrightarrow \mathbb{C}$ *be a class function. Then we have*

$$\int_{\mathrm{U}(n)} f = \frac{1}{(2\pi)^n n!} \int_0^{2\pi} \cdots \int_0^{2\pi} f\left(\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})\right) \prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|^2 \, d\theta_1 \ldots d\theta_n.$$

Proof. Let T be the maximal torus of $G = \mathrm{U}(n)$, consisting of digaonal matrices. For any function f on G, we have

$$\int_G f = \frac{1}{n!} \int_{T \times G/T} f(gtg^{-1}) J(t) \, dt \, d(gT),$$

where $J(t)$ is the Jacobian of the map $T \times G/T \to G$ given by $(t, gT) \mapsto gtg^{-1}$. When f is a class function this gives

$$\int_G f = K^{-1} \int_T f(t) J(t) dt,$$

where $K^{-1} = \mathrm{vol}(G/T)/n!$. Now, $J(t) = \det(\mathrm{Ad}(t^{-1}) - 1)$, where $\mathrm{Ad}(\cdot)$ is the adjoint action on $\mathfrak{t}^{\perp}$. The eigenvalues of $\mathrm{Ad}(t^{-1})$ are of the form $e^{i(\theta_k - \theta_j)}$, with eigenvectors given by the matrices E_{jk} which have all entries vanishing except the (j, k) entry which is equal to unity. So $J(t) = \prod_{j \neq k}(e^{i(\theta_k - \theta_j)} - 1) = \prod_{j<k} |e^{i\theta_k} - e^{i\theta_j}|^2$. Finally, we determine the constant $K = \int_T J(t) \, dt$. Note that $J(t)$ is equal to the square of the Vandermonde determinant, which is given by $\Delta(t) = \sum \pm e^{im_1\theta_1} \cdots e^{im_n\theta_n}$, where the sum runs over all $(m_1, \ldots, m_n)$ equal to the permutations of $(0, \ldots, n-1)$. There are $(n!)^2$ terms in the square, and they can be integrated each in turn, giving $K = (2\pi)^n n!$. $\square$

Remark. Almost the same calculation gives a similar result for any compact Lie group G with maximal torus T. The Jacobian in this case is equal to $|\Delta^+|^2$, where Δ^+ is the Weyl denominator (the denominator of the Weyl character formula). Notice that the Vandermonde determinant is the Weyl denominator for the group $\mathrm{U}(n)$.

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